MULTICRITERIA DECISION MAKING IN n-D[1]

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ABSTRACT: The paper concerns engineering design governed by multiple objective criteria that are in conflict and compete for available resources (material, financial, etc.). A multicriteria decision making (MCDM) strategy is presented that employs a tradeoff-analysis technique to identify compromise-design solutions that mutually satisfy the competing criteria in a Pareto-optimal sense. The concepts are initially illustrated in detail for a design governed by n=2 conflicting criteria. Curve-fitting, equation-discovery and equation-solving software are employed to find competitive general equilibrium states corresponding to Pareto-tradeoff designs of a flexural plate governed by conflicting weight and deflection criteria. The MCDM strategy is then extended to designs involving more than two conflicting criteria, and is applied for a bridge maintenance plan design governed by n=3 criteria. The paper concludes with a discussion of the application of the MCDM strategy to designs involving n=4 and n=11 conflicting criteria.

KEYWORDS: multicriteria design engineering, Pareto optimization, Pareto trade-off.

1 INTRODUCTION

Engineering design is generally governed by multiple conflicting criteria, which requires the designer to look for good compromise designs by performing tradeoff studies between them. As the competing criteria are often non-commensurable and their relative importance is generally not easy to establish, this suggests the use of non-dominated optimization to identify a set of designs that are equal-rank optimal in the sense that no design in the set is dominated by any other feasible design for all criteria. This approach is referred to as 'Pareto' optimization and has been extensively applied in the literature concerned with multicriteria engineering design (e.g., Osyczka 1984, Koski 1994, Khajehpour 2001, Grierson & Khajehpour 2002).

A Pareto optimization problem involving n conflicting objective criteria expressed as explicit or implicit functions $f_i(\boldsymbol{z})$ of design variables $\boldsymbol{z}$ ($i=1,2,\ldots,n$), can be concisely stated as:

$$\text{Minimize}\{\, f_1(z), f_2(z), \ldots, f_n(z) \,\};\ \text{Subject to } \mathbf{z} \in \Omega \qquad (1)$$

where $\boldsymbol{\Omega}$ is the feasible design space. A design $z^* \in \boldsymbol{\Omega}$ is a Pareto-optimal solution to the problem posed by Eq.(1) if there does not exist any other design $z \in \boldsymbol{\Omega}$ such that $f_i(z) \leq f_i(z^*)$ for $i=1,2,\ldots,n$ with $f_j(z) < f_j(z^*)$ for at least one criterion. The number of Pareto-optimal design solutions to Eq.(1) can be quite large, however, and it is yet necessary to select the best compromise design(s) from among them.

For example, consider the simply-supported plate with uniformly distributed loading shown in Figure 1. It is required to design the plate for the two conflicting criteria to minimize structural weight $f_1(\boldsymbol{z}) = W$ and midpoint deflection $f_2(\boldsymbol{z}) = \Delta$, for variables $\boldsymbol{z}$ taken as the thicknesses of pre-specified zones of the plate (see Koski 1994 for details). For any plate design z^*, its weight W^* is given by the explicit function $f_1(z^*)$ while its midspan deflection Δ^* is given by the implicit[2] function $f_2(z^*)$.

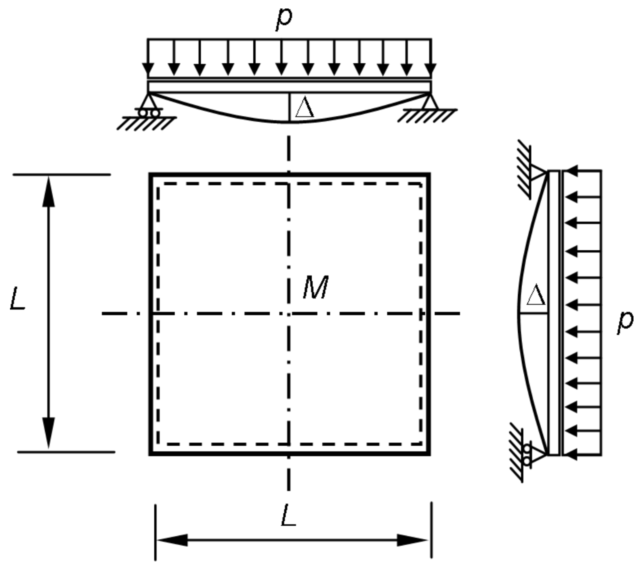

Figure 1. Flexural Plate - Loading & Deflection (Koski 1994).

Koski (1994) found ten Pareto-optimal designs having the weights $\boldsymbol{W}$ and deflections $\boldsymbol{\Delta}$ listed in columns 2 and 3 of

[1] n-dimensional Euclidean space

[2] $f_2(z^*) = \Delta^*$ implies deformation analysis of plate design z^* to find midpoint deflection Δ^*

Table 1. The ten Pareto designs define the *Pareto curve* in Figure 2; in fact, any one of the theoretically infinite number of points along this curve corresponds to a Pareto design. Therefore, it essentially remains to select a good-quality compromise plate design from among a theoretically infinite set of Pareto designs.

Table 1. Pareto Flexural Plate Designs (Koski 1994).

Pareto Design	$f_1(z)=W$ (kg)	$f_2(z)=\Delta$ (mm)	x (W/Wmax)	y (Δ/Δmax)	(1-x)	(1-y)
1	39.4	2.73	0.351	1.000	0.649	0.000
2	40.0	2.50	0.356	0.916	0.644	0.084
3	42.4	2.00	0.378	0.733	0.622	0.267
4	46.8	1.50	0.417	0.549	0.583	0.451
5	53.3	1.00	0.475	0.366	0.525	0.634
6	58.8	0.75	0.524	0.275	0.476	0.725
7	67.6	0.50	0.602	0.183	0.398	0.817
8	75.6	0.375	0.673	0.137	0.327	0.863
9	90.8	0.25	0.808	0.092	0.192	0.908
10	112.3	0.175	1.000	0.064	0.000	0.936

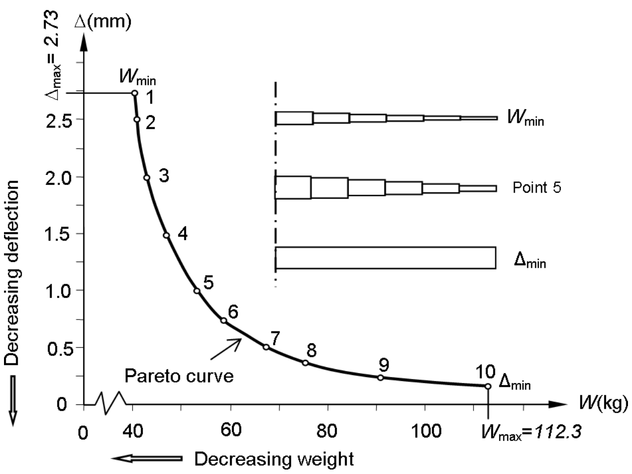

Figure 2. Pareto Flexural Plate Designs (Koski 1994).

The several methods proposed in the literature for searching among Pareto optima to select good-compromise designs are somewhat informal in that the selection process is primarily driven by designer preferences (see Koski 1994). Alternatively, a recent study by the author (Grierson 2006) developed a multicriteria decision making (MCDM) strategy adapted from the theory of social welfare economics (e.g., Boadway & Bruce 1984) that formally identifies competitive general equilibrium states corresponding to Pareto compromise designs; i.e., designs that represent a Pareto tradeoff between the competing criteria. The MCDM strategy is first reviewed in the following through reference to the two-criteria flexural plate design discussed in the foregoing (also see Grierson 2006). It is then extended to designs governed by any number n of conflicting criteria. The concepts are illustrated for a bridge maintenance plan design governed by $n=3$ conflicting criteria concerning bridge maintenance cost, condition and safety. Discussed is an office building design governed by $n=4$ conflicting criteria concerning building capital cost, life-cycle cost, revenue income and structural safety. The application of the MCDM strategy to a design governed by $n=11$ conflicting criteria is also briefly discussed.

2 TWO-DIMENSIONAL MULTICRITERIA DECISION MAKING

Consider a scenario in which two designers A and B are bargaining with each other to achieve an optimal tradeoff between $n=2$ competing criteria represented by two vectors of known values ($\boldsymbol{f_1}$, $\boldsymbol{f_2}$) found through Eq.(1) to define a set of Pareto designs for an engineered artifact (e.g., columns 2 and 3 of Table 1 for the flexural plate design). As the criteria are often non-commensurable and may have large differences in their numerical values, it is convenient to normalized their values as $\boldsymbol{x}= \boldsymbol{f_1}/f_1^{max}$ and $\boldsymbol{y}= \boldsymbol{f_2}/f_2^{max}$ (e.g., columns 4 and 5 of Table 1). With reference to the Pareto curve in Figure 2, for example, the corresponding *normalized Pareto curve* is as shown in Figure 3, where the maximum value for each of the two normalized criteria is unity.

Suppose that designer A is the advocate for the first criterion to minimize the (normalized) weight x and, therefore, that designer B is the advocate for the second criterion to minimize the (normalized) deflection y. Assume that designer A initially begins the bargaining session with the largest weight $x^{max}=1$, and that she considers making a tradeoff between the two criteria defined by the (absolute) value of the slope of the *terms-of-trade line* shown in Figure 3 passing through her initial point *(1,0)*. To that end, she would choose to trade at an intersection point of the trade line and the normalized Pareto curve so as to comply with the basic principles (structural, mechanical, financial, etc.) governing the feasibility of the Pareto designs. Moreover, if there is more than one such intersection point, as is the case in Figure3, designer A would choose to trade at that point for which the greatest decrease in weight occurs; i.e., she would trade at point E in Figure 3 by exchanging $1- x$ units of weight for y units of deflection. Before any such tradeoff can take place, however, the trading preferences of designer B must also be accounted for as in the following.

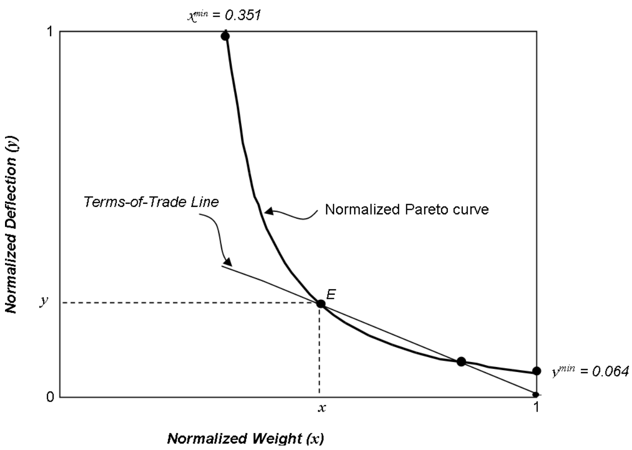

Figure 3. Two-Criteria Tradeoff.

We can draw a diagram similar to Figure 3 for designer B by supposing that he initially begins the bargaining session with the largest deflection $y^{max}=1$. Upon doing that, the competitive equilibrium of the two-designer and two-criteria tradeoff scenario can be analytically investigated by constructing the *Edgeworth-Grierson unit square*[3] (*E-G square)* in Figure 4. The origins for designers *A* and *B* are 0_A and 0_B, respectively (note that designer *B*'s axes are inverted since they are drawn with respect to origin 0_B). Their initial bargaining points *A(1,0)* and *B(0,1)* are both located at the lower right-hand corner of the unit square. Designer *A*'s Pareto curve PC_A is a plot of data points (*x*, *y*) in the fourth and fifth columns of Table 1, while designer *B*'s Pareto curve PC_B is a plot of data points (*1-x*, *1-y*) in the last two columns of Table 1.

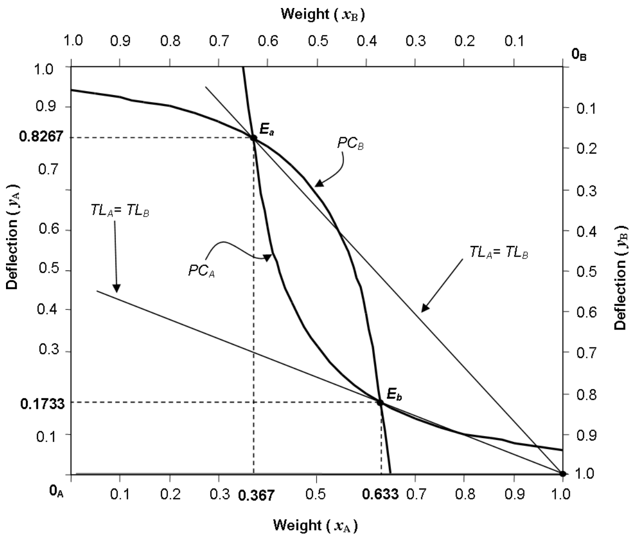

Figure 4: Edgeworth-Grierson unit square (Flexural plate design)

It is observed in Figure 4 that the Pareto curves PC_A and PC_B for designers *A* and *B* intersect at two points, E_a and E_b. Moreover, the terms-of-trade line through each intersection point is the same for both designers, i.e., $TL_A = TL_B$, which suggests the possibility for a mutually agreeable tradeoff at those points. In fact, points E_a and E_b are *competitive general equilibrium* states that each represent a *Pareto tradeoff* between the two competing criteria *x* and *y* (i.e., any movement away from points E_a and E_b will not result in a tradeoff state that is mutually agreeable to both designers).

The coordinates shown in Figure 4 for points E_a and E_b are found as follows. Upon applying curve-fitting/equation-discovery software (TableCurve2D 2005) to data points (*x*, *y*) in the fourth and fifth columns of Table 1, designer *A*'s Pareto curve PC_A is found to be accurately represented ($r^2 = 0.999$) by the function,[4]

$$17.15x^2y - 1.1y - 1 = 0 \quad (2)$$

Hence, from the last two columns of Table 1, designer *B*'s Pareto curve PC_B is represented by the function,

$$17.15(1-x)^2(1-y) - 1.1(1-y) - 1 = 0 \quad (3)$$

Upon applying simultaneous equation-solving software (MatLab 2005), Eqs. (2) and (3) are solved to find the two roots $(x_a^*, y_a^*) = (0.367, 0.827)$ and $(x_b^*, y_b^*) = (0.633, 0.173)$. That is, the *(x, y)* coordinates of the two equilibrium points are $E_1(0.367, 0.827)$ and $E_2(0.633, 0.173)$.

Equilibrium point E_a corresponds to a plate design intermediate to designs 2 and 3 in Table 1 that has weight $f_1^* = W^* = (0.367)(112.3) = 41.21kg$ and deflection $f_2^* = \Delta^* = (0.827)(2.73) = 2.26$ *mm,* while point E_b corresponds to a plate design intermediate to designs 7 and 8 in Table 1 that has weight $f_1^* = W^* = (0.633)(112.3) = 71.09$ *kg* and deflection $f_2^* = \Delta^* = (0.173)(2.73) = 0.472$ *mm.* While these two plate designs each represent a Pareto tradeoff between the competing weight and deflection criteria, they are not Pareto comparable between themselves. It yet remains for the designers to make a final selection between the two designs according to their preferences.

As the advocate for the weight criterion, designer *A* will opt for the plate design at point E_a because it has the least weight. However, as the advocate for the deflection criterion, designer *B* will alternatively prefer the plate design at point E_b because it has the least deflection. This dilemma is overcome if the two designers agree to act as a team that makes a compromise selection of one of the two designs. In effect, therefore, the MCDM strategy has served to significantly reduce the number of Pareto designs from which the final design selection is made based solely on designer preference (i.e., only two designs for this example).

3 PARETO DATA REQUIREMENTS

The MCDM tradeoff analysis depicted in Figure 4 implies the Pareto data $\boldsymbol{f_i} = [f_i^{min}, ..., f_i^{max}]^T$ for each competing criterion *i* satisfies certain conditions that ensure a competitive equilibrium point *E exists within* the boundary of the E-G square.

For an equilibrium point *E* to be *within* the boundary, it is *necessary* that f_i^{min} be greater than zero. This condition is naturally satisfied for most engineering criteria. If originally $f_i^{min} \leq 0$, as Pareto optimization is ordinal it is possible to make an additive uniform shift δ_i^+ of the floating-point data $\boldsymbol{f_i}$ to make $f_i^{min} + \delta_i^+ > 0$ without changing the Pareto nature of the data; i.e., uniformly add,

$$\delta_i^+ > |\, f_i^{min} | \qquad \{\text{if } f_i^{min} \leq 0 \text{ ; otherwise } \delta_i^+ = 0\} \quad (4)$$

For an equilibrium point *E* to *exist*, it is *sufficient* that the ratio f_i^{min}/f_i^{max} be less than or equal to $1 - \sqrt{2}/2 = 0.293$.[5] This condition is naturally satisfied for some engineering

[3] English economist F. Y. Edgeworth (1845-1926) was among the first to use a similar analytical tool known as the Edgeworth box to investigate the competitive equilibrium of a two-consumer and two-good exchange economy.

[4] Note that in Table 1 and Eqs.(2) & (3) the coordinates *x* and *y* are measured from the origin point 0_A in Figure 4 ; i.e., $x = x_A$ and $y = y_A$, and therefore $(1-x) = x_B$ and $(1-y) = y_B$.

[5] The limiting case when the Pareto curve is circular with radius $\sqrt{2}/2$, such that a single equilibrium point *E(0.5, 0.5)* exists at midpoint of the E-G square.

criteria. If originally $f_i^{min}/f_i^{max} > 1-\sqrt{2}/2$, as Pareto optimization is ordinal it is possible to make a subtractive uniform shift δ_i^- of the floating-point data $\boldsymbol{f_i}$ to make $(f_i^{min}-\delta_i^-)/(f_i^{max}-\delta_i^-) = 1-\sqrt{2}/2$ without changing the Pareto nature of the data; i.e, uniformly subtract,

$$\delta_i^- = f_i^{max} - \sqrt{2}(f_i^{max} - f_i^{min}) \quad \{\text{if } f_i^{min}/f_i^{max} > 0.293; \text{ otherwise } \delta_i^- = 0\} \tag{5}$$

From the foregoing, the existence of a competitive equilibrium point E within the boundary of the E-G square is ensured whenever the *original* or *shifted* Pareto data $\boldsymbol{f_i}=[f_i^{min},..., f_i^{max}]^T$ for each competing criterion i is such that,

$$0 < f_i^{min} \leq 0.293\, f_i^{max} \tag{6}$$

where the lower bound is a necessary and sufficient condition, while the upper bound is a sufficient condition.

That the upper bound in Eq.(6) is *not* a necessary condition is evidenced by the flexural plate example, for which the tradeoff analysis determined that two equilibrium points exist even though for the weight criterion the ratio $f_1^{min}/f_1^{max}=39.4/112.3=0.351> 0.293$ (see Table 1). However, the existence of equilibrium points in such circumstances depends on the shape of the Pareto curve and cannot be proved in general.

Whenever the original Pareto data $\boldsymbol{f_i}=[f_i^{min},..., f_i^{max}]^T$ for any criterion i does not satisfy the upper bound in Eq.(6), it is recommended that the data be shifted by uniformly subtracting δ_i^- defined by Eq.(5) so that Eq.(6) *is* satisfied. Then, after the MCDM tradeoff analysis is conducted to find each equilibrium point E and corresponding criteria values f_i^{**} ($i=1, 2$), the Pareto-tradeoff design value for each criterion i is found as,

$$f_i^* = f_i^{**} + \delta_i^- \tag{7}$$

For the flexural plate, for example, after shifting the Pareto data $\boldsymbol{f_1}$ for the weight criterion by uniformly subtracting $\delta_1^-=112.3-\sqrt{2}(112.3-39.4) =9.205$ *kg* (see Table 1)[6], the tradeoff analysis determines the two equilibrium points $E_a(0.305, 0.878)$ and $E_b(0.695, 0.123)$. Equilibrium point E_a corresponds to weight $f_1^{**}=(0.305)(112.3-9.205)=31.44$ *kg* and deflection $f_2^{**}=(0.878)(2.73)= 2.40$ *mm*, while point E_b corresponds to weight $f_1^{**}=(0.695)(112.3- 9.205)=71.65$ *kg* and deflection $f_2^{**}=(0.123)(2.73) = 0.336$ *mm*. Therefore, from Eq.(7), the Pareto-tradeoff plate design corresponding to point E_a is intermediate to designs 2 and 3 in Table 1 with weight $f_1^*= f_1^{**}+ \delta_1^- = 31.44+ 9.205 =40.65$ *kg* and deflection $f_2^*= f_2^{**}+ \delta_2^- = 2.40 + 0 = 2.40$ *mm*, while the Pareto-tradeoff design corresponding to point E_b is intermediate to designs 8 and 9 in Table 1 with weight $f_1^*= f_1^{**}+ \delta_1^- = 71.65 + 9.205 = 80.86$ *kg* and deflection $f_2^* = f_2^{**}+ \delta_2^- = 0.336 + 0 = 0.336$ *mm*.

It is observed for the flexural plate that the original and shifted Pareto-tradeoff designs at point E_a are almost identical (i.e., *41.21* versus *40.65 kg* weight, and *2.26* versus *2.40 mm* deflection), while those at point E_b are moderately different (i.e., *71.09* versus *80.86 kg* weight, and *0.472* versus *0.336 mm* deflection). In fact, it can be argued that the tradeoff design results are more for accurate for the shifted Pareto data as it is more representative of that part of the data which essentially determines its Pareto optimality.[7]

Finally, it is observed that it is not possible to shift the Pareto data for any criterion i for which $(f_i^{max}-f_i^{min})/f_i^{max}< \varepsilon$, where ε is the adopted tolerance for setting floating-point numerals to zero.[8] Such data is almost perfectly uniform, is not in meaningful conflict with the other objective criteria for the design, and can be assigned the fixed objective value $f_i^* = (f_i^{max} + f_i^{min})/2$ without affecting the remaining Pareto data set.

4 N-DIMENSIONAL MULTICRITERIA DECISION MAKING

The MCDM tradeoff strategy is generalized in the following to design problems governed by more than two conflicting criteria in competition for resources. Consider a design governed by $n > 2$ competing criteria represented by m-dimensional vectors $\boldsymbol{f_1}, \boldsymbol{f_2} ,..., \boldsymbol{f_n}$ of known values found through solution of Eq.(1) to define a Pareto set of m designs. The Pareto vectors are each normalized over the [0,1] range as $\boldsymbol{x_i}= \boldsymbol{f_i}/f_i^{max}$ ($i=1, 2,..., n$) to achieve the dimensionless and therefore commensurable data $\boldsymbol{x_1}, \boldsymbol{x_2},..., \boldsymbol{x_n}$.

By definition, a tradeoff can be made between only *two* criteria at any one time. For $n> 2$ criteria, this study investigates the tradeoff between each primary criterion and a corresponding aggregate criterion formed from the remaining n-1 criteria. The m-dimensional vectors $\boldsymbol{x_i}$ ($i=1, 2, ..., n$) are initially employed to create n pairs of vectors $(\boldsymbol{x_i}, \boldsymbol{y_i})$ where, for each pair, $\boldsymbol{x_i}$ is the vector of *primary criterion* values while $\boldsymbol{y_i}$ is a corresponding vector of *aggregate criterion* values found as,

$$\boldsymbol{y_i} = \prod \boldsymbol{x_j} \quad (j = 1, 2,..., n\ ;\ j \neq i) \tag{8}$$

For example, for a design problem governed by $n=3$ conflicting criteria defined by Pareto vectors $\boldsymbol{x_1}$, $\boldsymbol{x_2}$ and $\boldsymbol{x_3}$, evaluation of Eq.(8) for $i=1, 2, 3$ yields the following $n=3$

[6] Note that the Pareto data $\boldsymbol{f_2}$ for the deflection criterion is not shifted since, from Table 1, $f_2^{min}/f_2^{max} = 0.175/2.73 = 0.064 < 0.293$ and, therefore, $\delta_2^-= 0$ from Eq.(5).

[7] To put this statement in perspective, suppose a Pareto vector of original data for a financial objective criterion (e.g., minimize capital cost) consists of elements that are all between one and two million currency units (e.g., Dollar, Euro, etc.). One million currency units can be uniformly subtracted from all elements to create a Pareto vector of shifted data whose elements are all of the order of the thousands of currency units which determine the Pareto optimality of the original data.

[8] For example, $\varepsilon = 10^{-4} > 0.999x10^{-4} \approx 0$.

pairs of vectors: $(\boldsymbol{x_1}, \boldsymbol{y_1}) = (\boldsymbol{x_1}, \boldsymbol{x_2}^T\boldsymbol{x_3})$, $(\boldsymbol{x_2}, \boldsymbol{y_2}) = (\boldsymbol{x_2}, \boldsymbol{x_1}^T\boldsymbol{x_3})$ and $(\boldsymbol{x_3}, \boldsymbol{y_3}) = (\boldsymbol{x_3}, \boldsymbol{x_1}^T\boldsymbol{x_2})$.

Each $\boldsymbol{y_i}$ vector represents an aggregate criterion in conflict with a corresponding $\boldsymbol{x_i}$ vector representing a primary criterion. As the primary vectors $\boldsymbol{x_i}$ are normalized over the [0,1] range, it follows from Eq.(8) that the aggregate vectors $\boldsymbol{y_i}$ are similarly normalized and are thus commensurable among themselves and with the $\boldsymbol{x_i}$ vectors. However, even though the $\boldsymbol{y_i}$ vectors are formed from the Pareto set of $\boldsymbol{x_i}$ vectors, it does not follow that each pair of vectors $(\boldsymbol{x_i}, \boldsymbol{y_i})$ constitutes a Pareto set. As this is a necessary condition for application of the MCDM tradeoff strategy, a Pareto filter[9] is applied in turn to each of the n pairs of m-dimensional vectors $\boldsymbol{x_i}$ and $\boldsymbol{y_i}$ to retain a corresponding Pareto pair of reduced-dimension vectors $(\boldsymbol{x_i}, \boldsymbol{y_i})$, along with a record of the indices of the retained designs. As it is unlikely that the retained designs are the same for all n Pareto pairs, and as this is necessary to facilitate comparative interpretation of the results of the n tradeoff analyses, a design-index filter is further applied to retain only the $p< m$ designs that are common to all n Pareto pairs.[10] When necessary, $\boldsymbol{x_i}$ or $\boldsymbol{y_i}$ vector data is shifted by uniformly subtracting δ_i^- given by Eq.(5) so that Eq.(6) is satisfied (where, here, $f_i^{max}=x_i^{max}$ or y_i^{max}, and $f_i^{min}=x_i^{min}$ or y_i^{min}). Finally, where necessary, the p-dimensional $\boldsymbol{x_i}$ and $\boldsymbol{y_i}$ vectors are normalized as $\boldsymbol{x_i}= \boldsymbol{x_i}/x_i^{max}$ and $\mathbf{y}_i= \boldsymbol{y_i}/y_i^{max}$ to restore the data for all n Pareto pairs to the [0,1] range.

Having the $n> 2$ Pareto pairs of p-dimensional vectors $(\boldsymbol{x_i}, \boldsymbol{y_i})$, the MCDM tradeoff strategy is applied in turn to find for each vector pair i the two competitive general equilibrium points,

$$E_{ai}(x_{ai}^*, y_{ai}^*) \;\; ; \;\; E_{bi}(x_{bi}^*, y_{bi}^*) \qquad (i =1, 2,\dots , n) \qquad (9)$$

where values x_{ai}^* and x_{bi}^* of primary criterion i represent a Pareto tradeoff with values y_{ai}^* and y_{bi}^* of aggregate criterion i, respectively. It remains to select a final good-compromise design from among the $2n$ designs identified by points E_{ai} and E_{bi} (e.g., from among *six* designs if $n = 3$; see the following Bridge example).

5 BRIDGE MAINTENANCE PLAN DESIGN

It is required to design a bridge maintenance-intervention plan that exhibits optimal tradeoff between $n=3$ conflicting objective criteria concerning maintenance life-cycle cost, bridge condition, and bridge safety (Liu & Frangopol 2005). The life-cycle *cost criterion* involves minimization. The bridge *condition criterion* involves minimization, as it is represented by a damage-inspection index for which smaller values indicate better conditions. The *safety criterion* involves maximization, as it is represented by a load-capacity index for which larger values indicate more safety. The design is formulated as the Pareto optimization problem,

$$\text{Minimize}\{ f_1(z), f_2(z), f_3(z) \}; \text{ Subject to } z \in \boldsymbol{\Omega} \qquad (10)$$

where, from Eq.(1), z are the design variables and $\boldsymbol{\Omega}$ is the feasible design space. The function $f_1(z)$ = life-cycle cost, while $f_2(z)$ = condition index, and $f_3(z)$ = 1/(safety index)[11].

Liu and Frangopol (2005) solved Eq.(10) using a multicriteria genetic algorithm to find three *194x1* vectors $\boldsymbol{f_1}, \boldsymbol{f_2}, \boldsymbol{f_3}$ representing 194 Pareto designs of the bridge maintenance plan. The corresponding minimum and maximum criteria values, f_i^{min} and f_i^{max} $(i=1,2,3)$, are listed in Table 2.

Table 2. Pareto Min-Max Criteria Values (Liu & Frangopol 2005).

Criterion	f_i^{min}	f_i^{max}
Life-cycle Cost (k£) f_1	392.888	7009.637
Condition Index f_2	1.768	3.938
1/ (Safety Index) f_3	0.6106	0.8547

The MCDM strategy is applied to the 194 Pareto designs to identify a total of $2n=2x3=6$ Pareto-tradeoff designs, as follows:

1. For the f_i^{max} values in Table 2, normalize the *194x1* Pareto vectors $\boldsymbol{f_1}, \boldsymbol{f_2}, \boldsymbol{f_3}$ over the [0,1] range to create the *194x1* primary vectors $\boldsymbol{x_1}= \boldsymbol{f_1}/f_1^{max}$, $\boldsymbol{x_2}= \boldsymbol{f_2}/f_2^{max}$, $\boldsymbol{x_3}= \boldsymbol{f_3}/f_3^{max}$.
2. From Eq.(8), create the *194x1* aggregate vectors $\boldsymbol{y_1}= \boldsymbol{x_2}^T\boldsymbol{x_3}$, $\boldsymbol{y_2}= \boldsymbol{x_1}^T\boldsymbol{x_3}$, $\boldsymbol{y_3}= \boldsymbol{x_1}^T\boldsymbol{x_2}$.
3. Apply a Pareto filter to each of the $i=1, 2, 3$ pairs of *194x1* vectors $(\boldsymbol{x_i}, \boldsymbol{y_i})$, to create the three corresponding Pareto pairs of: *80x1* vectors $(\boldsymbol{x_1}, \boldsymbol{y_1})$; *49x1* vectors $(\boldsymbol{x_2}, \boldsymbol{y_2})$; *43x1* vectors $(\boldsymbol{x_3}, \boldsymbol{y_3})$.
4. Apply a design-index filter to the three variable-dimension Pareto pairs of vectors $(\boldsymbol{x_1}, \boldsymbol{y_1} ; \boldsymbol{x_2}, \boldsymbol{y_2} ; \boldsymbol{x_3}, \boldsymbol{y_3})$ created in Step 3, to create the three corresponding common-dimension Pareto pairs of *24x1* vectors $(\boldsymbol{x_1}, \boldsymbol{y_1} ; \boldsymbol{x_2}, \boldsymbol{y_2} ; \boldsymbol{x_3}, \boldsymbol{y_3})$.
5. For the *24x1* Pareto vectors $(\boldsymbol{x_1}, \boldsymbol{y_1} ; \boldsymbol{x_2}, \boldsymbol{y_2} ; \boldsymbol{x_3}, \boldsymbol{y_3})$ created in Step 4, calculate the following ratios and observe that vectors $\boldsymbol{x_2}$ and $\boldsymbol{x_3}$ do not satisfy the upper bound of Eq.(6):
   $x_1^{min}/x_1^{max}=0.056/0.974=0.057$,
   $y_1^{min}/y_1^{max}=0.087/0.977=0.089$

[9] A Pareto filter is a sorting algorithm based on the same principles as those governing the solution of the Pareto optimization problem posed by Eq.(1).

[10] It is important to note that $p<<m$; i.e., the two-tier filtering of the data significantly reduces the number of Pareto designs of concern to the MCDM analysis (e.g., 87% reduction for the Bridge example).

[11] Minimization of $1/f_i(z)$ is equivalent maximization of $f_i(z)$.

$x_2^{min}/x_2^{max}=0.450/0.994=0.453$,
$y_2^{min}/y_2^{max}=0.055/0.295=0.186$

$x_3^{min}/x_3^{max}=0.715/0.993=0.720$,
$y_3^{min}/y_3^{max}= 0.055/0.289=0.190$

6. From Eq.(5), uniformly subtract $\delta_2^- = 0.994-\sqrt{2}(0.994-0.450)=0.225$ from vector $\boldsymbol{x_2}$, and $\delta_3^- = 0.993-\sqrt{2}(0.993-0.715)=0.560$ from vector $\boldsymbol{x_3}$, to create two new *24x1* Pareto vectors $\boldsymbol{x_2}$ and $\boldsymbol{x_3}$ that identically satisfy the upper bound of Eq.(6).
7. For the x_i^{max} and y_i^{max} values from Steps 5 and 6, normalize the *24x1* vectors ($\boldsymbol{x_1}, \boldsymbol{y_1}$; $\boldsymbol{x_2}, \boldsymbol{y_2}$; $\boldsymbol{x_3}, \boldsymbol{y_3}$) created in Steps 4 and 6 over the [0,1] range, to create the Pareto primary-aggregate criteria pairs of *24x1* vectors $\boldsymbol{x_i}, \boldsymbol{y_i}$ $(i=1, 2, 3)$ listed in Table 3 along with the indices of the corresponding 24 designs retained from among the original 194 Pareto designs.
8. Apply curve-fitting/equation-discovery software (TableCurve2D 2005) for each of the three pairs of Pareto vectors ($\boldsymbol{x_i}, \boldsymbol{y_i}$) in Table 3, to find that each of the three corresponding Pareto curves is accurately represented ($r^2 \geq 0.988$) by the function,

$$c_i\, x_i\, y_i + d_i\, y_i - 1 = 0 \qquad (i=1,2,3) \qquad (11)$$

where $c_1=13.231$, $c_2=5.710$ and $c_3=5.611$, while $d_1=0.198$, $d_2=-0.624$ and $d_3=-0.634$.
9. As for the E-G square, formulate the inverse function,

$$c_i\,(1-x_i)(1-y_i) + d_i\,(1-y_i) - 1 = 0 \qquad (i=1,2,3) \qquad (12)$$

10. Apply simultaneous equation-solving software (MatLab 2005) to solve Eqs. (11) and (12), to find for each primary-aggregate criteria pair *i* the two competitive general equilibrium points,

$$E_{ai}(\, x_{ai}^*\, , y_{ai}^*\,) \;\; ; \;\; E_{bi}(\, x_{bi}^*\, , y_{bi}^*\,) \;\; (\, i = 1, 2, 3) \qquad (13)$$

where:

$x_{a1}^*=0.0672$, $y_{a1}^*=0.9203$; $x_{b1}^*=0.9328$, $y_{b1}^*=0.0797$
$x_{a2}^*=0.3743$, $y_{a2}^*=0.6609$; $x_{b2}^*=0.6257$, $y_{b2}^*=0.3391$
$x_{a3}^*=0.3912$, $y_{a3}^*=0.6405$; $x_{b3}^*=0.6088$, $y_{b3}^*=0.3595$

11. To complete the MCDM analysis, account for the normalization parameters f_i^{max} and x_i^{max} used in Steps 1 and 7, respectively, and the shift parameters δ_i^- used in Step 6, to relate the six primary criteria values x_{ai}^*, x_{bi}^* $(i=1,2,3)$ found in Step 10 to the six Pareto-tradeoff bridge maintenance plan designs f_1^*, f_2^*, f_3^* listed in Table 4. Figure 5, consisting of three E-G squares, provides a geometrical interpretation of the MCDM analysis.

The design indices 34, 54, 69, 78, 84 and 179 indicated in Table 4 and Figure 5 refer to the six designs from among the original 194 Pareto designs that are closest to the Pareto-compromise design points defined by Eq.(13); i.e., six bridge maintenance plan designs that represent a Pareto tradeoff between the three competing objective criteria to minimize life-cycle maintenance cost, minimize bridge damage condition, and maximize bridge safety. It yet remains for the designers to make a final selection from among the six designs according to their preferences.

Table 3: Pareto Pairs of Primary-Aggregate Criteria for Bridge Maintenance Plan Design

Design Index	Primary $x_1 = Cost$	Aggregate $y_1 = x_2^T x_3$	Primary $x_2 = Condition$	Aggregate $y_2 = x_1^T x_3$	Primary $x_3 = Safety$	Aggregate $y_3 = x_1^T x_2$
24	0.225	0.304	0.569	0.388	0.528	0.431
34	0.556	0.137	0.342	0.715	0.391	0.642
39	0.617	0.125	0.337	0.736	0.362	0.702
52	0.199	0.349	0.591	0.380	0.585	0.396
63	0.120	0.564	0.764	0.288	0.735	0.309
82	0.281	0.236	0.502	0.427	0.463	0.475
84	0.067	0.923	0.988	0.204	0.934	0.223
87	0.213	0.320	0.565	0.390	0.561	0.404
98	0.225	0.303	0.574	0.385	0.522	0.435
108	0.849	0.097	0.300	0.885	0.315	0.862
121	0.058	1.000	1.000	0.187	1.000	0.193
123	0.105	0.620	0.807	0.261	0.765	0.283
125	0.081	0.799	0.896	0.236	0.890	0.245
127	0.162	0.398	0.664	0.314	0.595	0.360
130	0.269	0.252	0.492	0.446	0.506	0.446
132	0.118	0.568	0.766	0.283	0.739	0.302
133	0.092	0.741	0.869	0.254	0.851	0.267
134	0.323	0.202	0.452	0.467	0.440	0.491
135	0.578	0.129	0.365	0.656	0.344	0.714
144	0.993	0.092	0.293	1.000	0.304	0.985
158	1.000	0.089	0.295	0.970	0.293	1.000
159	0.111	0.614	0.805	0.274	0.760	0.300
171	0.340	0.199	0.450	0.485	0.435	0.515
194	0.074	0.873	0.949	0.221	0.919	0.235

Table 4. Pareto-Tradeoff Bridge Maintenance Plans (Liu & Frangopol 2005).

Design Index	Life-cycle Cost (k£) f_1^*	Condition Index f_2^*	Safety Index f_3^* $[1/f_3^*]$
34	3797.126	1.920	0.644 [1.553]
54	1207.650	2.785	0.714 [1.401]
69	3938.305	2.005	0.640 [1.563]
78	1178.656	2.799	0.717 [1.395]
84	459.043	3.881	0.827 [1.209]
179	6732.955	1.796	0.613 [1.631]

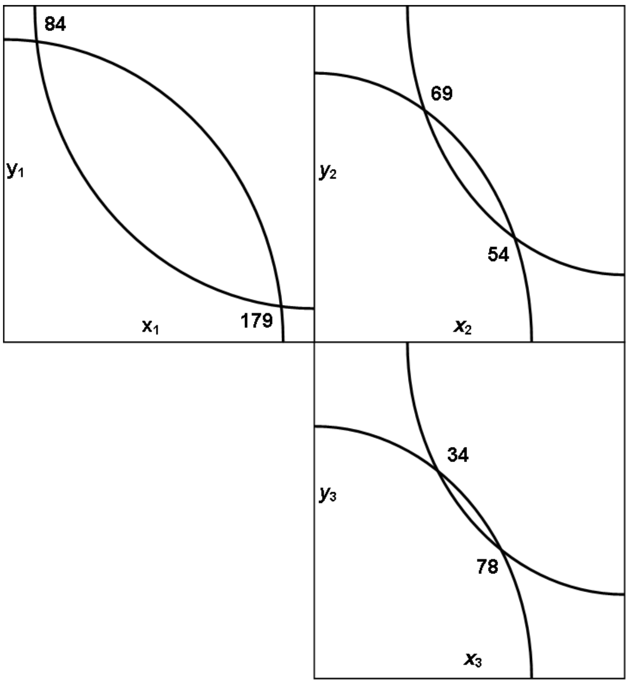

Figure 5. Edgeworth-Grierson tromino[12] (Bridge maintenance plan design).

[12] Three squares connected at their edges.

6 PENDING APPLICATIONS OF THE 'MCDM' STRATEGY[13]

It is intended to design a multi-story office building that exhibits optimal tradeoff between $n=4$ conflicting objective criteria concerning capital cost, life-cycle cost, income revenue and structural safety. The capital cost and life-cycle cost criteria involve minimization, while the revenue and safety criteria involve maximization. The design can be formulated as the Pareto optimization problem,

$$\text{Minimize}\{\, f_1(\mathbf{z}), f_2(\mathbf{z}), f_3(\mathbf{z}), f_4(\mathbf{z}) \,\};\ \text{Subject to } \mathbf{z} \in \boldsymbol{\Omega} \quad (14)$$

where z are the design variables and Ω is the feasible design space. The function $f_1(z)$ = capital cost, while $f_2(z)$ = life-cycle cost, $f_3(z)$ = 1/(revenue) and $f_4(z)$ = 1/(safety). Khajehpour and Grierson (2003) solved a similar problem to Eq.(14) using a multicriteria genetic algorithm to find four $815x1$ vectors $\boldsymbol{f_1}$, $\boldsymbol{f_2}$, $\boldsymbol{f_3}$, $\boldsymbol{f_4}$ representing 815 Pareto designs of the of the office building. It yet remains to identify the $2n=2x4=8$ Pareto tradeoff-compromise designs of the building; i.e., eight building designs from among the 815 Pareto designs that represent a Pareto tradeoff between the four competing objective criteria to minimize capital and life-cycle costs and maximize revenue and safety.

It is intended to design a media centre that exhibits optimal tradeoff between $n=11$ conflicting objective criteria concerning building cost and lighting performance. Four of the criteria involve minimization and seven involve maximization. Shea *et al* (2006) recognized that $4.2x10^{298}$ possible designs exist, and applied a multicriteria ant colony optimization method with Pareto filtering to find a large number of Pareto designs. It yet remains to identify the $2n=2x11=22$ Pareto tradeoff-compromise designs of the media centre; i.e., twenty-two Pareto designs that represent a Pareto tradeoff between the eleven competing objective criteria concerning cost and lighting.

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REFERENCES

Boadway, R., and Bruce, N. (1984). *Welfare Economics*. Basil Blackwell, Chapter 3, 61-67.

Grierson, D. E. (2006). "Welfare Economics Applied to Design Engineering", Proceedings of the 13th EG-ICE Workshop on *Intelligent Computing in Engineering and Architecture*, Edited by I. F. C. Smith, Ascona, Switzerland, June 25-30, 298-314.

Grierson, D. E., and Khajehpour, S. (2002). "Method for Conceptual Design Applied to Office Buildings." *J. of Computing in Civil Engineering*, ASCE, N. Y., 16 (2), 83-103.

Khajehpour, S., and Grierson, D. E. (2003) "Profitability versus Safety of High-rise Office Buildings", Journal of Structural & Multidisciplinary Optimization, Springer, 25,1-15.

Khajehpour, S. (2001) *Optimal Conceptual Design of High-Rise Office Buildings*. PhD Thesis, Civil Engineering, University of Waterloo, Canada, pp 191.

Koski, J. (1994). "Multicriterion Structural Optimization." *Advances in Design Optimization*, Edited by H. Adeli, Chapman and Hall, New York, Chapter 6, 194-224.

Liu, M., and Frangopol, D. M. (2005). "Bridge Annual Maintenance Prioritization under Uncertainty by Multiobjective Combinatorial Optimization", *Computer-Aided Civil and Infrastructure Engineering*, Blackwell Publishing, 20, 343-353.

Osyczka, A. (1984). *Multicriterion Optimization in Engineering*. Ellis Horwood, Chichester

Shea, K., Sedgwick, A., and Antonuntto, G. (2006). "Multicriteria Optimization of Paneled Building Envelopes Using Ant Colony Optimization", Proceedings of the 13th EG-ICE Workshop on *Intelligent Computing in Engineering and Architecture*, Edited by I. F. C. Smith, Ascona, Switzerland, June 25-30,627-636.

MATLAB, Version 7.0 (2005). *Automated Equation Solver*. The MathWorks, Inc.

TableCurve2D, Version 5.01 (2005). *Automated Curve-fitting and Equation Discovery*. Systat Software, Inc., CA.

[13] To be presented at the 2007 Maribor workshop.