PARETO ANALYSIS OF PARETO DESIGN

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ABSTRACT

The paper concerns design engineering problems involving multiple criteria and, in particular, the development of a formal trade-off strategy that can be employed by designers to mutually satisfy conflicting criteria as best as possible. A Pareto-optimal exchange analysis technique is adapted from the theory of social welfare economics as the basis for a search methodology to identify good-quality compromise designs. The concepts are presented for the two-criteria design problem so that the main ideas can be given a geometric interpretation. Curve-fitting, equation-discovery and equation-solving software are employed along with welfare economics analysis to find competitive general equilibrium states corresponding to Pareto-optimal compromise designs of a flexural plate governed by conflicting weight and deflection criteria.

KEY WORDS

multi-criteria design engineering, multi-goods welfare economics, Pareto optimization.

INTRODUCTION

One of the difficulties in engineering design is that there are generally several conflicting criteria, which forces the designer to look for good compromise designs by performing trade-off studies between them. As the conflicting criteria are often non-commensurable and their relative importance is generally not easy to establish, this suggests the use of non-dominated optimization to identify a set of designs that are equal-rank optimal in the sense that no design in the set is dominated by any other feasible design for all criteria. This approach is referred to as 'Pareto' optimization and has been extensively applied in the literature concerned with multi-criteria engineering design (e.g., Koski 1994, Grierson and Khajehpour 2002). The number of Pareto-optimal designs so found can still be quite large, however, and it is yet necessary to select the best compromise design(s) from among them.

Koski (1994) briefly reviews several methods for searching among Pareto optima to identify one or more good compromise designs. The final selection generally depends on the designer's personal preferences. This study employs a Pareto trade-off analysis technique adapted from the theory of social welfare economics to identify one or more competitive general equilibrium states of the conflicting criteria that represent good compromise designs; i.e., designs that represent a Pareto-optimal compromise between designer preferences for the various criteria. The trade-off strategy is developed for the two-criteria problem so that the concepts can be given a geometric interpretation. A flexural plate design involving conflicting weight and deflection criteria serves to illustrate the main ideas.

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WELFARE ECONOMICS

UTILITY FUNCTIONS

In the theory of welfare economics involving multiple goods, utility functions are used to describe consumer preferences for different bundles of the goods. For an economy involving two goods, x_1 and x_2, a utility function assigns a number to every possible consumption bundle (x_1,x_2) such that more-preferred bundles get assigned larger numbers than less-preferred bundles. The utility assignment is 'ordinal' in that it serves only to *rank* the different consumption bundles, while the size of the utility difference between any two bundles isn't meaningful.

In the space of the two goods x_1 and x_2, a certain consumption bundle (x_1,x_2) lies on the boundary of the set of all bundles that the consumer perceives as being preferred to it. This implies that the consumer is indifferent to all bundles that lie on the set boundary itself, which is called an *indifference curve*. Utility functions are used to define (label) indifference curves such that those which are associated with greater preferences get assigned higher utility numbers. They can take on a variety of forms (a specific utility function form is considered later in the paper).

For any incremental changes dx_1 and dx_2 of the two goods along an indifference curve defined by utility function $u(x_1,x_2)$, there is no change in the value of the utility function. Mathematically, this may be expressed as (Varian 1992),

$$du = \frac{\partial u(x_1,x_2)}{\partial x_1}dx_1 + \frac{\partial u(x_1,x_2)}{\partial x_2}dx_2 = 0 \qquad (1)$$

Equation (1) may be reorganized to find the slope of the indifference curve as,

$$\frac{dx_2}{dx_1} = -\frac{\partial u(x_1,x_2)/\partial x_1}{\partial u(x_1,x_2)/\partial x_2} \qquad (2)$$

which is known as the *marginal rate of substitution (MRS)* that measures the rate at which the consumer is willing to substitute good x_2 for good x_1. The negative sign indicates that if the amount of good x_1 *increases* then the amount of good x_2 *decreases* in order to keep the same level of utility, and vice versa.

PARETO EXCHANGE & COMPETITIVE EQUILIBRIUM

Consider now a pure exchange economy in which two consumers *A* and *B* are seeking to achieve an optimal trade-off between goods x_1 and x_2 (Boadway and Bruce 1984). The total supply of good x_1 is x_1^*, while that for good x_2 is x_2^*. Suppose that consumer *A*'s initial endowment consists of the entire supply of good x_1, as indicated by the distance between the origin 0_A and point x_1^* along the horizontal axis in Figure 1. Her initial utility level is u_A^0. If consumer *A* is offered a relative price of good x_1 in terms of good x_2 as given by the (absolute) value of the slope of the *terms-of-trade line* (TL_A) passing through her endowment point x_1^*, she will choose to trade $x_1^*-x_{1A}$ units of good x_1 in exchange for x_{2A} units of good x_2 and, thereby, achieve increased utility level u_A^1 (note that utility increases with distance from the origin 0_A).

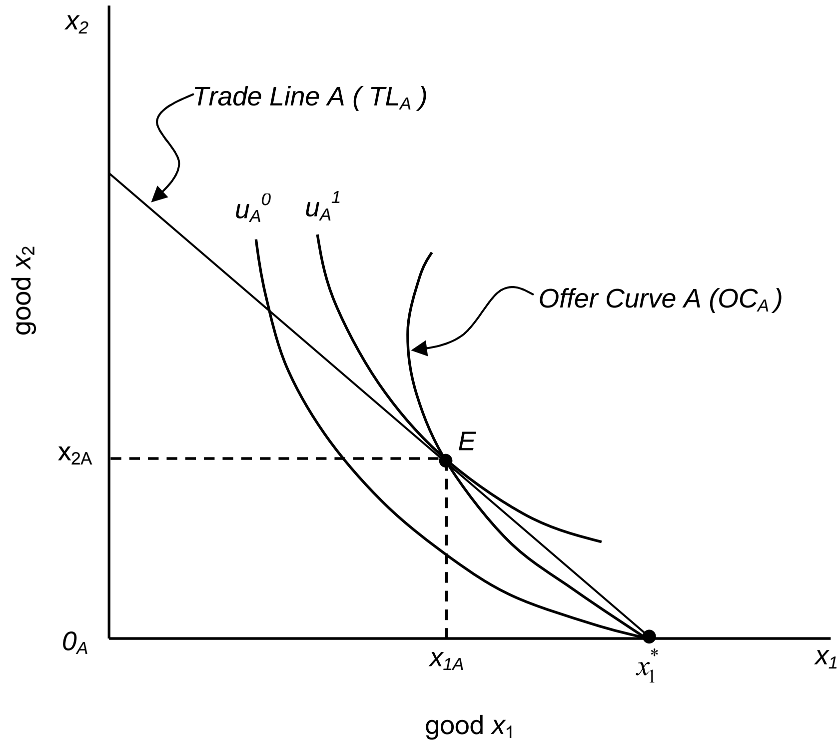

Figure 1: Two-Good Exchange Economy (Boadway and Bruce 1984)

By offering consumer *A* different relative prices of good x_1 in terms of good x_2, her *offer curve* (OC_A) can be traced out by rotating the terms-of-trade line through her endowment point and drawing the locus of equilibrium points *E* chosen. The highest indifference curve (utility) available to consumer *A* at each relative price is tangent to the terms-of-trade line at its intersection with the offer curve. Offer curves indicate the willingness of consumers to exchange a certain amount of one good for a given amount of another good at any relative price. They can take on a variety of forms (a specific offer curve is considered later in the paper).

We can draw a similar diagram for consumer *B* by supposing that his initial endowment consists of the entire supply of good x_2. Upon doing that, the competitive equilibrium of the two-consumer and two-good exchange economy can be analytically examined by constructing the *Edgeworth box*[2] diagram in Figure 2, the horizontal and vertical dimensions for which are equal to the total supplies x_1^* and x_2^* of goods x_1 and x_2, respectively. The origins for consumers *A* and *B* are 0_A and 0_B, respectively. Their initial endowment points $A(x_1^*, 0)$ and $B(0, x_2^*)$ are both located at the lower right-hand corner of the box (note that consumer *B*'s axes are inverted since they are drawn with respect to origin 0_B).

2 Named in honor of English economist F. Y. Edgeworth (1845-1926), who was among the first to use this analytical tool.

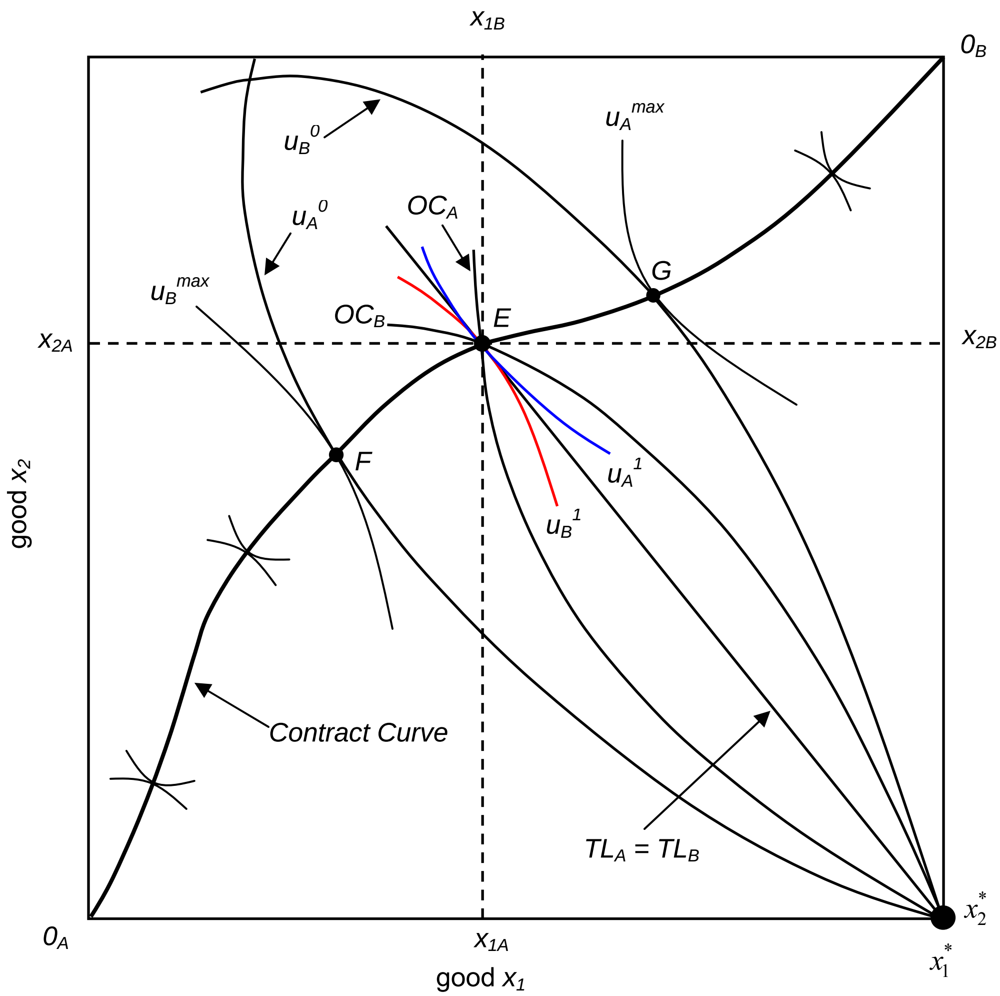

Figure 2: Welfare Economics Edgeworth Box (Boadway and Bruce 1984)

Of special interest is the *contract curve*[3], which is the locus of all allocations of the two goods such that the indifference curves of consumer *A* are tangent to those of consumer *B*. That is, the marginal rate of substitution between goods x_1 and x_2 for consumer *A* is equal to that for consumer *B* at each point on the contract curve, but not at any other point off the curve. This suggests the possibility for mutually beneficial trade.

The initial indifference curves u_A^0 and u_B^0 shown in Figure 2 form a lens-shaped area within which lie points that are Pareto superior to the initial endowment point and which can be reached by consumers *A* and *B* through trading goods x_1 and x_2 between themselves. Once they have traded to a point on the contract curve no further Pareto improvements are possible, since then one consumer can gain utility only at the expense of the other. That is, any point on the contract curve segment *FG* is a Pareto-optimal allocation of goods x_1 and x_2

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[3] Each point on the contract curve is obtained by maximizing the utility of one consumer while holding that for the other consumer fixed: e.g., point *G* in Figure 2 is found by *maximizing* $u_A(x_{1A},x_{2A})$ *subject to* $u_B(x_{1B},x_{2B})=u_B^0$, $x_{1A}+x_{1B}=x_1^*$ *and* $x_{2A}+x_{2B}=x_2^*$.

between consumers *A* and *B*. But some points are better than others depending on the consumer; namely, all points from *F* up to almost *E* are unacceptable to consumer *A* because they do not lie on her offer curve and have smaller utility than desired, while the same situation applies for consumer *B* for all points from *G* up to almost *E*. It is only at the intersection point *E* of their offer curves OC_A and OC_B that consumers *A* and *B* are mutually satisfied with their highest attainable utilities u_A^1 and u_B^1, respectively. Point *E* is a *competitive general equilibrium* Pareto-optimal allocation of goods x_1 and x_2.

That point *E* lies on the contract curve follows from the fact the two offer curves at that point have the same marginal rate of substitution. In other words, as indicated in Figure 2, there exists a common terms-of-trade line $TL_A = TL_B$ that affords consumers *A* and *B* the opportunity to trade from their initial endowment to point *E*. This opportunity to directly proceed to a Pareto-optimal competitive general equilibrium state is exploited in the following concerning multi-criteria design engineering.

DESIGN ENGINEERING

FLEXURAL PLATE DESIGN

Consider the simply-supported plate with uniformly distributed loading shown in Figure 3, for which length L = 600 mm, load P =0.4 N/mm^2, material density ρ = 7800 kg/m^3, Young's modulus E_{ym} = 206×10^3 N/mm^2, and Poisson's ratio ν = 0.3 (Koski 1994). It is required to design the plate for the two conflicting criteria of *minimum weight (W-criterion),* and *minimum deflection (Δ-criterion)* at midpoint *M*.

The analysis model for the plate is defined by the mesh of 36 finite elements shown in Figure 4(a). The plate thicknesses of the six zones indicated in the design model for the plate shown in Figure 4(b) are taken as the design variables. The (von Mises) stress σ_i (i =1, 2, …, 36) for each finite element is constrained to be less than or at most equal to 140 MPa, while the thickness t_j (j =1, 2, …, 6) for each plate zone is constrained to be in the range of 2-40 mm.

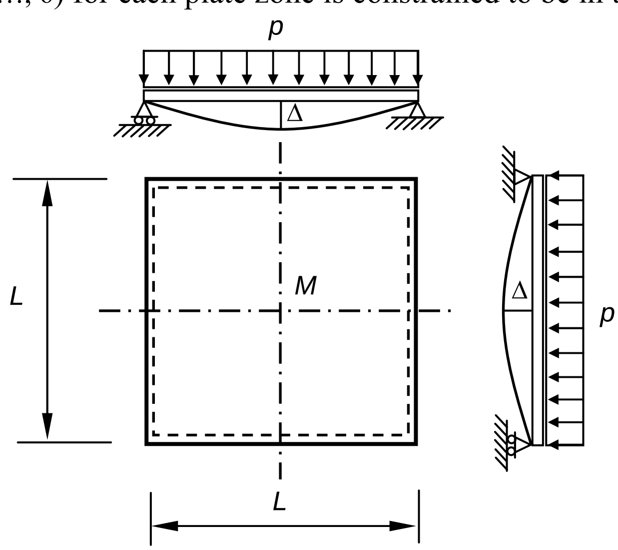

Figure 3: Flexural Plate - Loading & Deflection (Koski 1994)

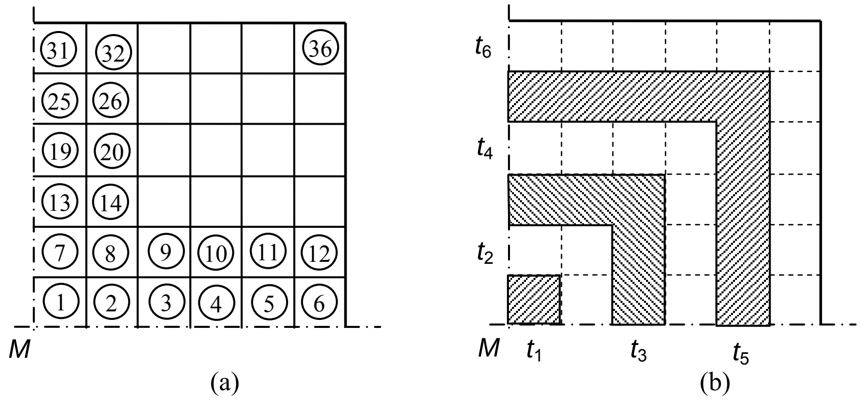

Figure 4: Quarter-Plate (a) Analysis Model, (b) Design Model (Koski 1994)

The 2-criteria design optimization problem statement is:

$$\left.\begin{array}{ll} \text{Minimize: } & F(\underline{t}) = [W(\underline{t}), \Delta(\underline{t})] \\ \text{Subject to: } & \sigma_i \leq 140 \qquad (i = 1,2,...,36) \\ & 2 \leq t_j \leq 40 \qquad (j = 1,2,...,6) \end{array}\right\} \tag{3}$$

A variety of optimization methods are available to find a Pareto-optimal design set for the problem posed by Eqs.(3). A genetic algorithm could be applied for solution (Grierson and Khajehpour 2002). Alternatively, Koski (1994) used sequential quadratic programming to find the ten Pareto-optimal designs having variously different plate thicknesses listed in Table 1. It can be seen from the last two table columns that no one design is dominated by any other design for both the W-criterion and the Δ-criterion. The ten Pareto-optimal designs define the *Pareto curve* in Figure 5; in fact, any point along this curve corresponds to a Pareto design. Shown are sketches of the Pareto designs corresponding to W_{min}, point *5* and Δ_{min}. It remains to select a good-quality compromise design from among the set of Pareto designs in accordance with the preferences of the design team.

Table 1: Pareto-Optimal Flexural Plate Designs (Koski 1994)

Design Point	t_1 (mm)	t_2 (mm)	t_3 (mm)	t_4 (mm)	t_5 (mm)	t_6 (mm)	W (kg)	Δ (mm)
1	20.6	19.7	18.4	16.4	13.8	8.6	39.4	2.73
2	26.1	20.8	18.4	16.4	13.8	8.6	40.0	2.50
3	30.2	26.1	20.6	16.4	13.8	8.6	42.4	2.00
4	31.0	28.9	24.7	19.4	14.1	8.6	46.8	1.50
5	37.3	34.3	26.8	22.1	16.3	9.8	53.3	1.00
6	40.0	37.1	30.2	24.0	18.3	10.8	58.8	0.75
7	40.0	40.0	36.4	27.8	21.0	12.8	67.6	0.50
8	40.0	40.0	40.0	32.6	24.6	14.4	75.6	0.375
9	40.0	40.0	40.0	40.0	33.5	20.5	90.8	0.25
10	40.0	40.0	40.0	40.0	40.0	40.0	112.3	0.1746

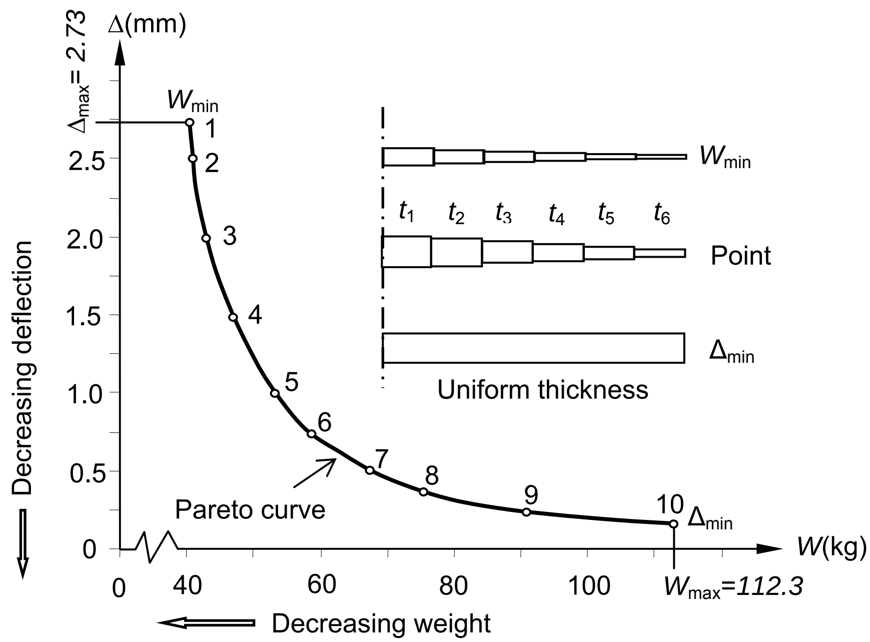

Figure 5: Pareto-Optimal Flexural Plate Designs (Koski 1994)

PARETO-OPTIMAL DESIGN COMPROMISE

A formal trade-off strategy based on the welfare economics analysis[4] presented earlier is applied in the following to identify a compromise plate design for which designer preferences concerning the conflicting W and Δ criteria are Pareto optimal (in fact, two such designs are identified).

To begin, normalize the data for the W and Δ criteria in the last two columns of Table 1 to be as given by the x_1 and x_2 values in the fourth and fifth columns of Table 2. Note that the largest value of each of the normalized criteria x_1 and x_2 is unity (i.e., $x_1^* = x_2^* = 1.0$). Then consider two designers A and B who are seeking between themselves to achieve an optimal trade-off of the two criteria x_1 and x_2 for the plate design. Suppose that designer A's initial endowment is the largest value $x_1^* = 1.0$ of criterion x_1, while that for designer B is the largest value $x_2^* = 1.0$ of criterion x_2.

Similar to that in Figure 2, the competitive equilibrium of the two-designer and two-criterion trade-off exercise can be analytically examined by constructing the normalized Edgeworth box diagram in Figure 6, the horizontal and vertical dimensions for which are both equal to unity. The origins for designers A and B are 0_A and 0_B, respectively, and their initial endowment points $A(1,0)$ and $B(0, 1)$ are both located at the lower right-hand corner of the box. Designer A's offer curve OC_A is a plot of the data points (x_1,x_2) in the fourth and fifth columns of Table 2 (i.e., a normalized plot of the Pareto curve in Figure 5), while designer B's offer curve OC_B is a plot of the data points $(1\text{-}x_1, 1\text{-}x_2)$ in the last two columns of Table 2.

[4] Here: consumers ≡ designers; goods ≡ criteria; $x_1 \equiv W$-criterion; $x_2 \equiv \Delta$-criterion

Table 2: Data for Flexural Plate Design Trade-Off Analysis

Design Point	Weight (W)	Deflection (Δ)	x_1 (W/Wmax)	x_2 (Δ/Δmax)	$(1-x_1)$	$(1-x_2)$
1	39.4	2.73	0.351	1.000	0.649	0.000
2	40.0	2.50	0.356	0.916	0.644	0.084
3	42.4	2.00	0.378	0.733	0.622	0.267
4	46.8	1.50	0.417	0.549	0.583	0.451
5	53.3	1.00	0.475	0.366	0.525	0.634
6	58.8	0.75	0.524	0.275	0.476	0.725
7	67.6	0.50	0.602	0.183	0.398	0.817
8	75.6	0.375	0.673	0.137	0.327	0.863
9	90.8	0.25	0.801	0.092	0.199	0.908
10	112.3	0.1746	1.000	0.064	0.000	0.936

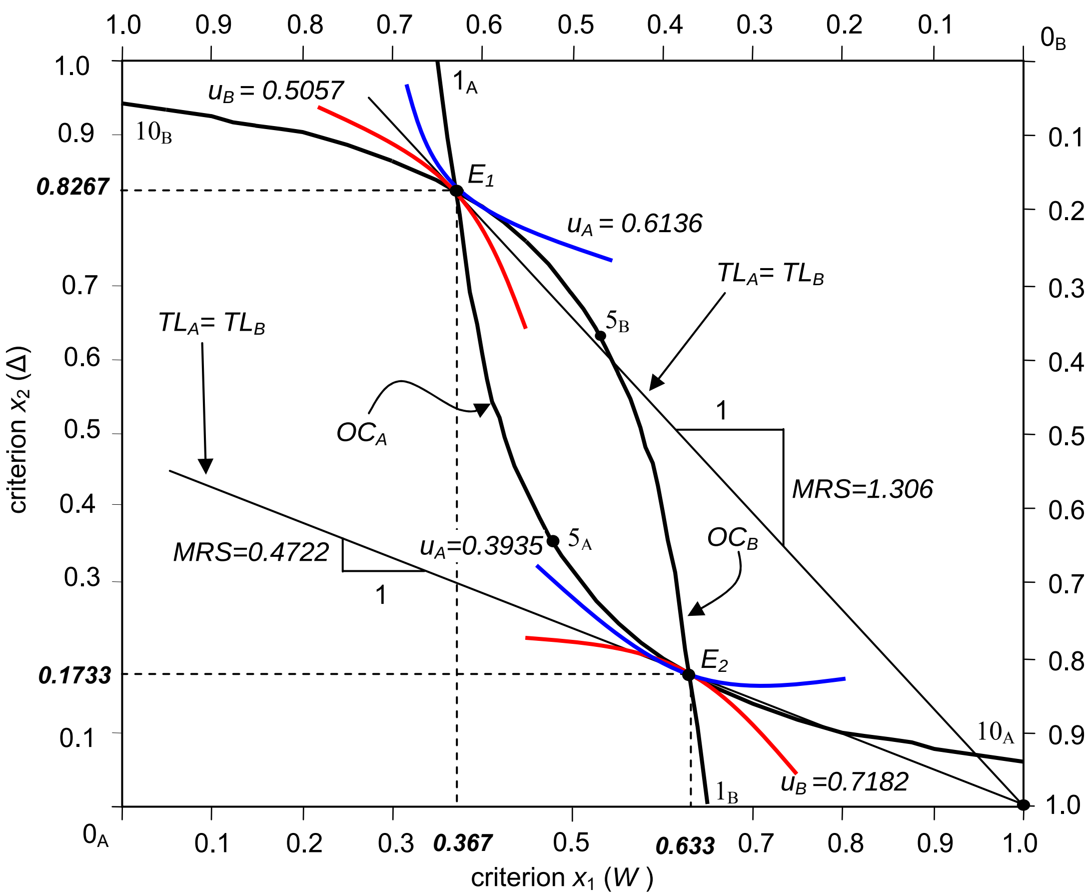

Figure 6: Design Engineering Edgeworth Box

Competitive General Equilibrium

It is observed in Figure 6 that there are *two* competitive general equilibrium points E_1 and E_2 defined by the intersections of the two offer curves OC_A and OC_B, the coordinates for which are found as follows. Upon applying curve-fitting/equation-discovery software to the data points (x_1,x_2) in the fourth and fifth columns of Table 2, designer *A*'s offer curve OC_A is found to be accurately represented ($r^2 = 0.999$) by the function (TableCurve2D v5.01),[5]

$$17.15x_1^2x_2 - 1.1x_2 - 1 = 0 \tag{4}$$

Hence, designer *B*'s offer curve OC_B is represented by the function,

$$17.15(1-x_1)^2(1-x_2) - 1.1(1-x_2) - 1 = 0 \tag{5}$$

Substitute for x_2 from Eq.(4) into Eq.(5) to obtain the function,

$$9.021x_1^4 - 18.042x_1^3 + 6.812x_1^2 + 2.209x_1 - 1 = 0 \tag{6}$$

Equation (6) is solved to find the meaningful roots $x_1 = 0.367$ and $x_1 = 0.633$ (MATLAB 7.0), and then the corresponding roots $x_2 = 0.8267$ and $x_2 = 0.1733$ are found through Eq.(4). That is, as shown in Figure 6, the equilibrium points are $E_1(0.367, 0.8267)$ and $E_2(0.633, 0.1733)$.

Design Utility

Equilibrium point E_1 corresponds to a plate design intermediate to designs 2 and 3 in Table 1 that has weight $W = (0.367)(112.3) = 41.21\ kg$ and deflection $\Delta = (0.8267)(2.73) = 2.26\ mm$, while point E_2 corresponds to a plate design intermediate to designs 7 and 8 in Table 1 that has weight $W = (0.633)(112.3) = 71.09\ kg$ and deflection $\Delta = (0.1733)(2.73) = 0.473\ mm$. Each of these two designs is a Pareto-optimal compromise design of the plate. It remains to determine their utilities from the perspective of the designers' preferences for the two conflicting criteria.

This study adopts the commonly used *Cobb-Douglas* utility function (Varian 1992),

$$u(x_1,x_2) = x_1^c x_2^{1-c} \tag{7}$$

where the exponent *c* is a function of the point at which the utility is being measured. Upon observing in Figure 6 that for both designers the marginal rate of substitution at any point (x_1,x_2) is $MRS = x_2/(1-x_1)$, is it found from Eqs. (2) and (7) that the utility functions for designers *A* and *B* can be expressed as,

$$u_A(x_1,x_2) = x_1^{x_1}x_2^{1-x_1} \quad ; \quad u_B(x_1,x_2) = (1-x_1)^{x_2}(1-x_2)^{1-x_2} \tag{8a,b}$$

The utility levels u_A and u_B indicated in Figure 6 are found by evaluating Eqs.(8) for the two sets of (x_1, x_2) coordinates ($0.367, 0.8267$) and ($0.633, 0.1733$) corresponding to equilibrium points E_1 and E_2, respectively. As expected, utility u_A is greater at E_1 than at E_2 (i.e., $0.6136 > 0.3935$) because the plate weight *W* there is less (i.e., $41.21\ kg < 71.09\ kg$). Conversely, utility u_B

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[5] Note that in Table 2 and Eqs.(4)-(8) the coordinates x_1 and x_2 are measured from the origin point 0_A in Figure 6; i.e., $x_1 = x_{1A}$ and $x_2 = x_{2A}$, and therefore $(1-x_1) = x_{1B}$ and $(1-x_2) = x_{2B}$.

is greater at E_2 than at E_1 (i.e., 0.7182 > 0.5057) because the plate deflection Δ there is less (i.e., 0.473 mm < 2.26 mm). Presuming that designer *A* is the advocate for the *W*-criterion, she will opt for the plate design at point E_1 because it provides her greatest utility $u_A = 0.6136$. However, as the advocate for the Δ-criterion, designer *B* will alternatively opt for the plate design at point E_2 because it provides his greatest utility $u_B = 0.7182$. This poses a dilemma, which may be overcome if the two designers agree to act together as a team that simply opts for the design having the maximum utility level u_{max} from all among all four utility levels associated with the two equilibrium points. That is, they would select the plate design at point E_2 having weight $W = 71.09\ kg$, deflection $\Delta = 0.473\ mm$, and utility $u_{max} = 0.7182$.

CONCLUDING REMARKS

This research is in its early stages and at present prompts fewer conclusions than it does questions, some of which are as follows. Does the form of the utility function have an influence on the results? When there are multiple general equilibrium points, what is the veracity of taking the solution to be that particular equilibrium point having the maximum utility level from all among all utility levels associated with all equilibrium points? Can the methodology be applied to design problems involving three or more conflicting criteria? These and other lines of enquiry will be pursued by the ongoing research program.

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