CHORDAL SPACE STRUCTURES, SHAPED FROM VORONOI DIAGRAMS

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ABSTRACT

The definition of the shape of the enclosures of large spaces by means of large span structures, where the absence of intermediate supports is a strong conditioning factor, is an interdisciplinary activity where geometry, biology, topology, architecture and engineering have complemented themselves. A relatively recent discipline, COMPUTATIONAL GEOMETRY, has permitted a new formulation of the geometric basis and the numerical procedures that permit to generate a spatial dome, regardless of its type.

The geometric and topologic configuration of any Spatial Mesh or Structure (including typologies like Lattice, Geotangent or any other patented or published structural form) does not suppose anything else than the creation of a polyhedron that approximates the shape of the ideal surface. Thanks to the methods of Computational Geometry, we are able to demonstrate that this problem has a purely and exclusively two-dimensional nature and treatment.

KEYWORDS

Computational Geometry, Spatial Structures, Voronoi Diagrams, Chordal Space Structures, Structural Morphology

INTRODUCTION. DEFINITION AND TYPOLOGY OF SPACE STRUCTURES

A space frame is a structural system assembled from linear elements so arranged that forces are transferred in a three-dimensional manner. In some cases, the constituent elements may be two-dimensional. Macroscopically a space frame often takes the form of a flat or curved surface (Tsuboi,1984). That classical definition can be extended according to the following classification of space structures (Wester,1990):

- Lattice archetype: frames composed by bars (one-dimensional elements) interconnected at nodes (zero-dimensional point objects). The structure is stabilized by the interaction of axial forces that concur at the nodes (fig.1, left).

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- Plate archetype: plates (bi-dimensional elements) that conform a polyhedron's faces stabilized by the shear forces acting along its edges (one-dimensional hinges) (fig. 2, left).
- Solid archetype: structures composed by three-dimensional elements which are stabilized by the action of forces transferred between the planar faces of the solids.

GEOMETRIC GENERATION OF SPACE STRUCTURES

The design of space structures can be approached from different points of view. We now review three methods developed during the second half of the 20th century that suggest different ways for approximating a quadric surface taken as reference.

Geodesic Dome: lattice type structure with a configuration derived from regular or semi-regular polyhedra in which the edges are subdivided into equal number of parts ("frequency", Fuller, 1954); making use of these subdivisions, a three-way grid can be induced upon the faces of the original polyhedron. The central projection of these grid's vertices on the polyhedron's circumsphere (see fig. 1), leads to a polyhedron approximating the sphere in which only the lattice's nodes lie on the sphere's surface (Kitrick,1990).

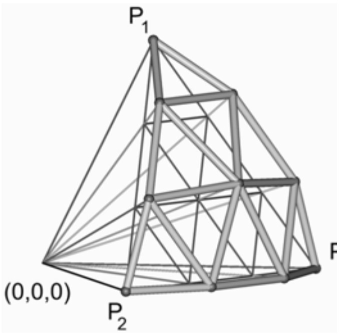

Fig. 1. *Left*: Generation of the Geodesic Dome through the projection of the three-way grid on the circumscribed sphere. *Right*: U.S. Pavilion, Montreal Universal Exposition (1967).

Geotangent Dome: a plate type polyhedral structure in which the edges are tangent to a sphere. Such a sphere is sectioned by the polyhedron's faces in such a way (fig. 2, above) that the faces' inscribed circles are tangent to the inscribed circles of neighboring faces. Following this rule it is possible to determine the planes containing the circles generating the polyhedron's edges from their intersection (Yacoe, 1987). The procedure is involved and its calculations imply the solution of a non-linear equation system through an iterative process based on successive approximations.

Panel Structure: these plate type structures (Wester, 1990) derive from lattice type geometries by applying the principle of structural and geometric duality (based on the concept of a point's polarity regarding a sphere). Taking as a starting point the geodesic dome's circumsphere, it is possible to transform the lattice's nodes in the faces of its dual structure (fig. 2, below); the primitive sphere remains as the new structure's insphere.

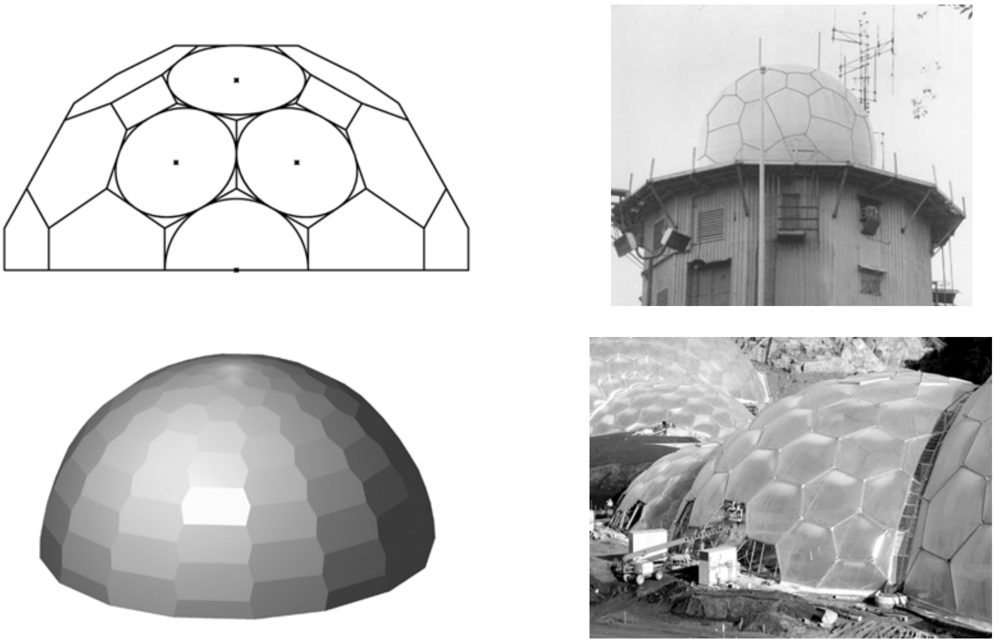

Fig. 2. Above: (*left*) Geotangent Polyhedron elevation. (R*ight*). Nine meter diameter geotangent dome crowning Canopy Tower, Cerro Semáforo, Panamá (1963). Below: (*left*) Panel structure, derived as the dual polyhedron of a Schwedler type dome. (R*ight*) Structures suggesting the plate typology. Eden Project, Cornwall, UK.

COMPUTATIONAL GEOMETRY: VORONOI DIAGRAM AND DELAUNAY TRIANGULATION ON THE PLANE

Voronoi Diagrams and Delaunay Triangulations belong to Computational Geometry, a modern field of study that can supply some new ideas to structural design. We will make a brief introduction of Voronoi Diagram and Delaunay Triangulation in plane 2D (Preparata, 1985). Let us consider a set S of points in the plane, $S = \{S_1, S_2, \ldots S_n\}$, where $S_i = (x_i, y_i)$. Given a member S_i of this set, the polygon of Voronoi of S_i, $V(S_i)$, is defined as the set of points P_i on the plane that are closer to S_i than to any other point S_j of S. A Voronoi polygon is shown in figure 3. Each point S_i of S defines a Voronoi polygon; the n regions arising from the set S partition the plane into a convex net which is known as the Voronoi Diagram of S (see figure 3). Each line segment of the diagram is a Voronoi edge, and its end points are called Voronoi Vertices. Notice that the S_i points are not vertices of the diagram: we will refer to them as Generator Points. Next, we list a number of important properties of the Voronoi Diagram: (i) every Voronoi Polygon $V(S_i)$ contains only one S_i generator point , (ii) every $V(S_i)$ is a convex polygon, (iii) every edge of the Voronoi Diagram is the perpendicular bisector of two generator points S_i and S_j, (iv) Every vertex of Voronoi is the common intersection of exactly three edges of the diagram. Only when four generator points are cocircular, does the related Voronoi vertex join 4 edges, (v) the straight-line dual of the Voronoi Diagram is a triangulation of S. (This means a planar subdivision of the plane, where every polygon is a triangle and where each vertex is a generator point; see fig. 3).

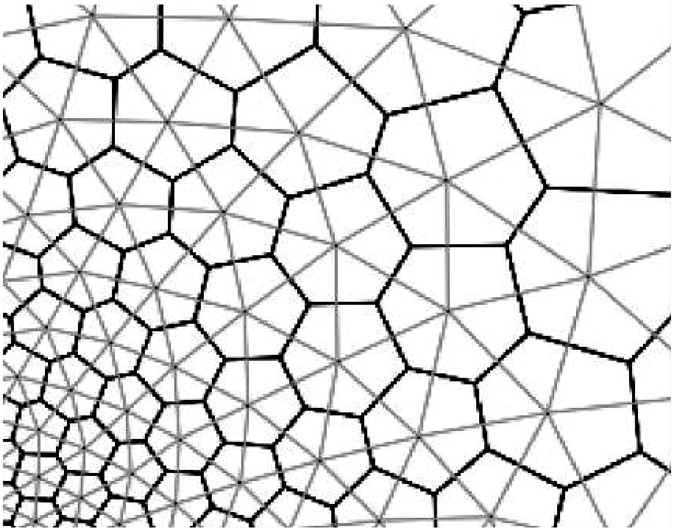

Fig. 3. Voronoi Diagram, Delaunay Triangulation and their duality.

SOME USEFUL PROPERTIES OF THE VORONOI DIAGRAM. AN APPLICATION TO THE GEOMETRY OF SPATIAL STRUCTURES.

Let then S'={P'_1, P'_2, . . . , P'_n} be a set of points in the plane z=1 and let VD be its related Voronoi Diagram. Let us consider an inversive transformation on R^3, with center on O(0,0,0) and power k=1; this transformation maps the plane z=1 into the sphere *E* $[x^2+y^2+(z-1/2)^2=1/4]$ and the set S' into another one S ={P_1, P_2, . . . , P_n}, were any P_i belongs to the sphere E. We have demonstrated (Otero, 2000) that:

PROP. 1. *The mapped image of the Voronoi Diagram of S' is a polyhedron that approximates the sphere E* $[x^2+y^2+(z-1/2)^2=1/4]$ *in such a way that each one of its faces is tangential to the sphere. There is a symmetric correspondence between each Voronoi polygon and each face of the polyhedron*. Note that the vertices of these polyhedron's faces do not belong to the sphere.

PROP 2. *The projection from the point of coordinates* $(0,0, 1/(2Z_c-1))$ *of the Voronoi Polygons of S' onto the second order surface SF* $[X^2 + Y^2 - Z^2(1 - 2Z_c) - 2ZcZ + 1 = 0]$ *makes up an approximating polyhedron the faces of which are tangent to SF. Each point of the set S' is transformed to the point of contact between the face of the polyhedron and the surface. The edges of the polyhedron are those transformed from the edges of the Voronoi Diagram of S': (i) When zc =1/2, SF is a rotated paraboloid, (ii) when zc <1/2, SF is a rotated (two folds) hyperboloid, (iii) when zc >1/2, SF is a rotated ellipsoid*.

PROP2-BIS. *It is enough to move the sphere E anywhere, but keeping it tangent to the plane z=1, to obtain, by the way stated in Prop. 2, polyhedra approximating non-revolution quadrics* (Otero, 2002).

A 2D PROCEDURE FOR CREATING MESHES MADE UP BY NONTRIANGULAR FACES

We have arrived at a very different family of approaches to the sphere, where the property of tangency is not produced on the edges of the polyhedron but in their faces. This supposes a more intuitive way for choosing the final shape of the body because its topology can be proposed by means of a 2D Voronoi Diagram at the plane z=1. We can handle different hypotheses easily, as we illustrate in the figure 4. On the other hand, all the properties are valid when the Voronoi Diagram is replaced by its dual transformation, the Delaunay Triangulation DT. So, not only panel structures, but lattice structures too, can be generated from a simple set of points on z=1, being possible to shape different types of quadric domes.

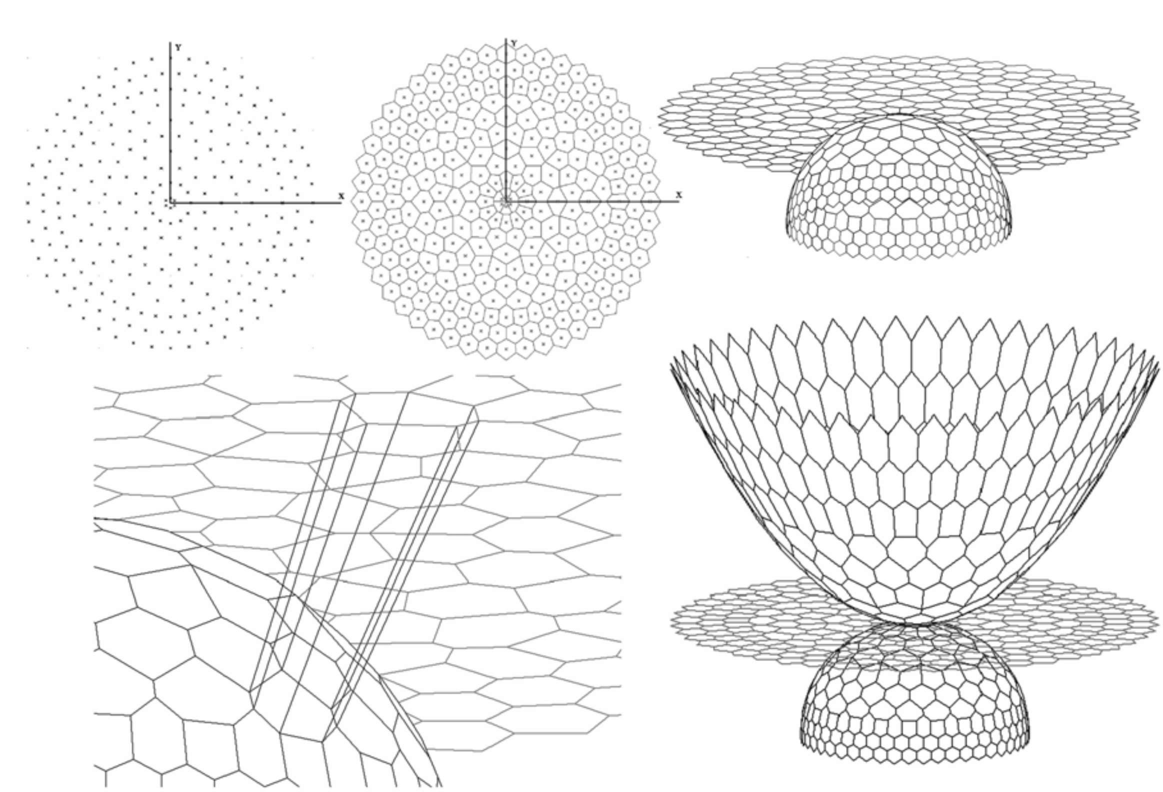

Fig. 4. (Above): *(left)* a set of point on the plane z=1, *(center)* makes automatically arise a Voronoi Diagram, VD, on this plane; *(right)* an inversive transformation with center on (0,0,0) maps this VD into a polyhedron circumscribed to the sphere with center on (0,0, ½) and radius ½. (Below): (left) each face on VD is mapped into a face on the polyhedron, keeping that each edge of VD is an edge on the aforementioned polyhedron. (Right) A proyective transformation maps each face of the polyhedron into new one, approaching different types of quadric surface.

POWER DIAGRAMS

Given a collection of circumferences lying on a plane (Togores, 2003), it is well known that a Planar Division derived from this set exists: it is obtained by the intersection of the power lines for each pair of properly chosen neighboring circles (fig. 5, left). To each circumference, a convex region of the plane is associated (Aurenhammer, 1991), which is defined by the intersection of half planes containing those points with the least circle power. This region is known as the power cell, and the set of cells for the said collection of circumferences is known as its associated power diagram.

The Power Diagram of a set of circles can be handled in a similar manner than we have done with the Voronoi Diagram in the previous points (Díaz, 2004). The result can be resumed in the next properties.

PROP 3*: The contact curve of the cone circumscribed to a quadric from an exterior point P is the conic section generated by the polar plane of point P. If among all the possible quadrics we select the paraboloid Ω: [*$z = x^2 + y^2$*], it is also true that the orthogonal projection of this section on any horizontal plane z=K is a circle.*

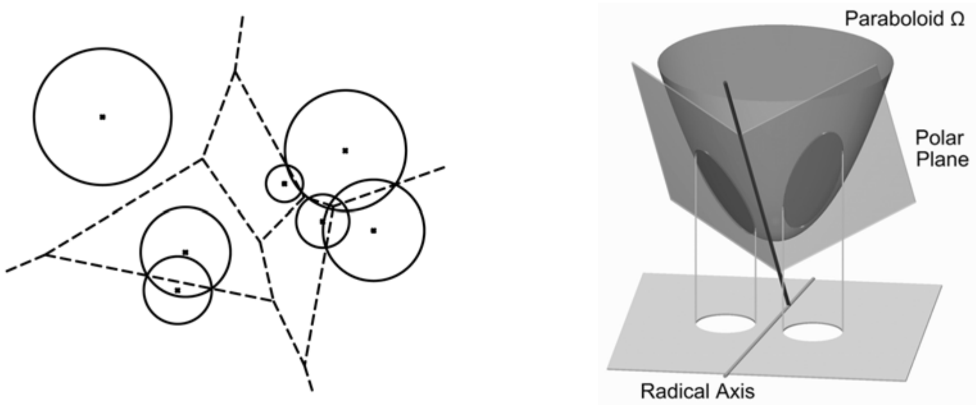

Fig. 5. (Left) Power Diagram of a set of 7 circumferences. (Right) Spatial interpretation of a chordale

PROP 4: *Given two points P and Q outside of the paraboloid Ω, it results then that their respective polar planes: (i) generate on Ω two ellipses that are projected on z=1 as two circles; (ii) the intersection of these polar planes is projected as the power line (radical axis) of the aforementioned circles.* See figure 5, right.

An immediate consequence is that:

PROP 5: E*very power diagram is the equivalent of the orthogonal projection of the boundaries of a convex polyhedral surface (resulting from the intersection of the half-spaces defined by polar planes)*. This surface can be regarded as a polyhedron that approximates the quadric.

DESIGN OF CHORDAL SPACE STRUCTURES

These last properties provide us with a mechanism to associate a cloud of points in space with the faces of a polyhedron approximating the paraboloid Ω; the obtained polyhedral body can be partially inscribed, partially circumscribed, partially tangent to its edges and partially secant to it (fig. 6). Again, it is demonstrated (Díaz, 2004) that not only for the paraboloid Ω, but for any other not ruled quadric this procedure is valid and useful.

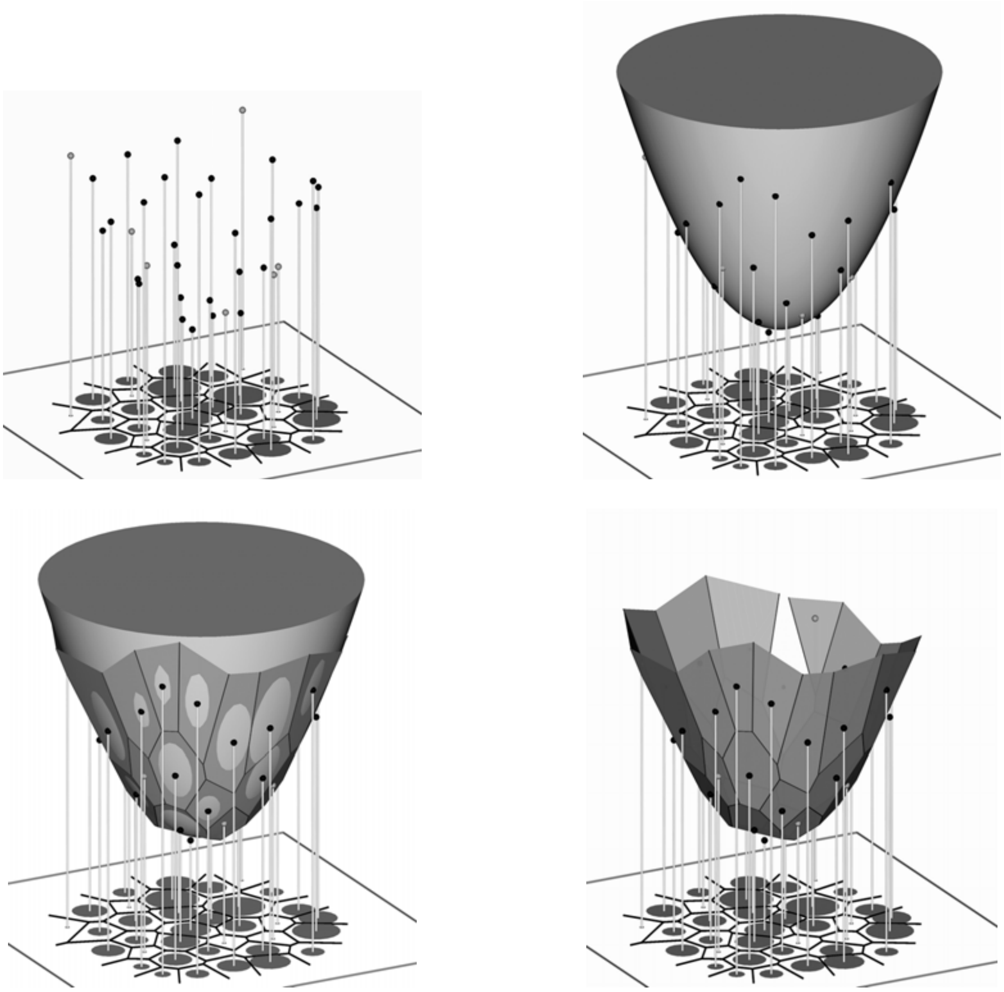

Fig. 6. A one-to-one correspondence ($C^2 \rightarrow E^3$) as the mechanism of definition of the polyhedron's faces that approximate the quadric. The relative positions of points with respect to Paraboloid Ω conditions the typology of the resulting structure, which can be predicted from the associated power diagram.

The field of knowledge related with the processes by which spatial structures are obtained is plagued with innumerable typologies, procedures, classes, subclasses and patents that Computational Geometry can synthesize in one single category: that we have named as **Chordal Space Structures**. This proposal simplifies and widens the scope of this technical activity. Nothing like this has been claimed before, because the intimate relation between Computational Geometry and the design of big lightweight structures remained unnoticed. Both fields are representative of progress in the XXth century and can go forward hand in hand in the XXIst.

GLOBULAR STRUCTURES AND FLATTENED STRUCTURES

Let us consider now the "parallel trihedra" procedure defined in Alvaro (2000). It is not difficult to notice that: (i) the operation of adding parallel trihedra around a center is feasible starting from both regular and semi-regular polyhedrons; and, (ii) the spherical patches will prove to be more convex or tauter (Fig. 7) than the sphere's surface circumscribing the initial polyhedron according to the position of point O_i in relation to the only degree of freedom that has this composition of patches of sphere (see the reference for more detail).

Fig. 7. Surfaces resulting from the aggregation of parallel trihedrons according to an icosahedral symmetry: globular (left) and flattened (right).

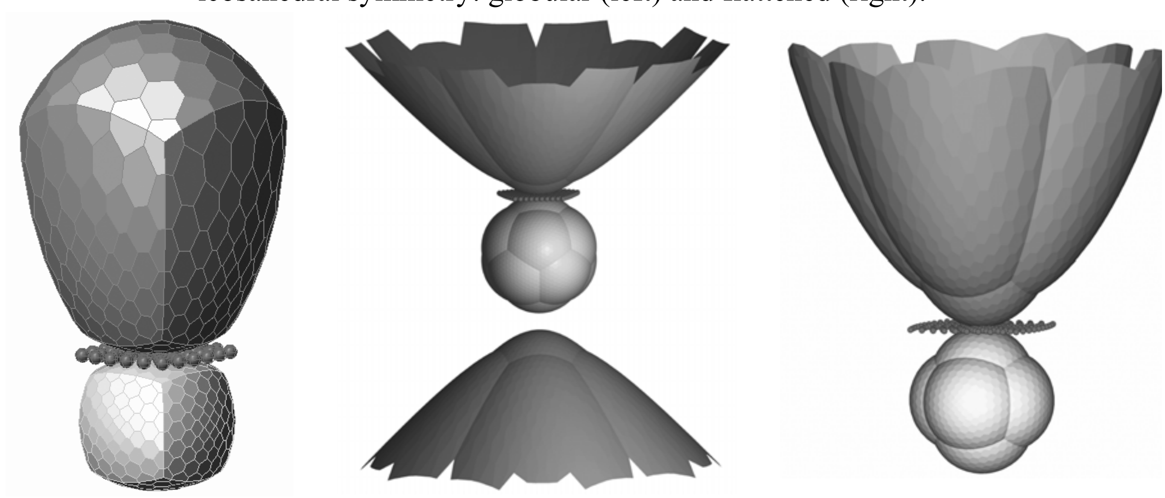

Fig.8. Examples of flattened (left) and globular meshes (center and right).

Resorting to the previous paragraph, it is possible to expand the catalog of Chordal Structures with forms that we shall call Flattened or Globular (see Fig. 8): the resulting quadric (ellipsoid, hyperboloid or paraboloid), as well as the intermediate polyhedron approximating the sphere is included.

Nevertheless, as engineers, our interest will not lie so much in approximating whole quadrics, but in portions of them that will permit us to define, for example, space enclosures. As a conclusion, we now include some illustrative examples (see Fig. 9). But at the end, it is important to not forget the main feature of the process: *the designer just selected a single set of points in a plane!!*

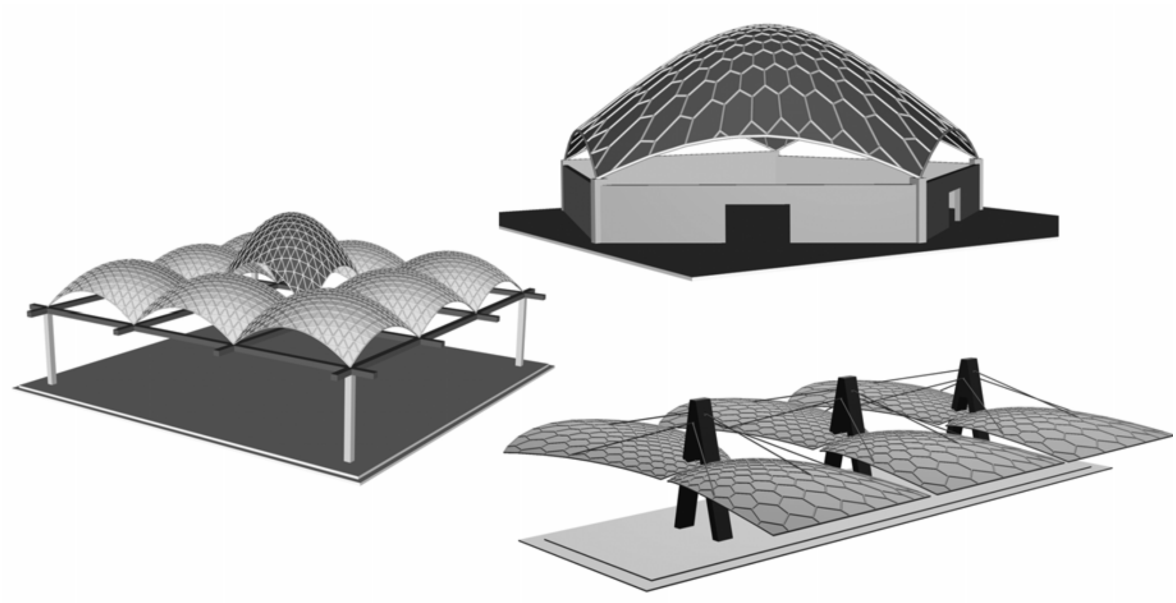

Fig. 9. Portions of quadrics used as space enclosures.

CONCLUSION

In the last years, a special effort is being made by the authors in order to spread out the potentialities of Computational Geometry in relation to the design of Spatial Structures. Under this premise, a central idea animates this article: the adequate interpretation in the physical three-dimensional space of the mathematical formulation that relates power diagrams with certain polyhedral structures is enough for promoting a novel way to approach the always complicated process of definition of space meshes. This is not the only way of relating the circumferences lying in a plane with the points and planes in space, but being so simple and intuitive, and due to the fact that it is developed from well established procedures, it results in a contribution of an especial interest for everyone that has in mind a research in the realm of the space enclosures.

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