Cornish-Fisher Expansion for Real Estate Value at Risk

Charles-Olivier Amédée-Manesme
THEMA, Université de Cergy-Pontoise, 33, Bd du Port, 95011, Cergy-Pontoise Cedex, France.
Tel : (33) 6 64 16 47 25. Email : charleso.amedeemanesme@gmail.com

Fabrice Barthélémy
THEMA, Université de Cergy-Pontoise, 33, Bd du Port, 95011, Cergy-Pontoise Cedex, France.
Tel: (33) 1 34 25 62 53. Email: fabrice.Barthélémy@eco.u-cergy.fr

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EARLY DRAFT VERSION

Abstract

Value at risk is a convenient and popular risk measurement tool. It represents the maximum potential loss on a specific portfolio of financial assets given a specific time horizon and a confidence interval. Principally value at risk is used in finance for risk management, financial reporting and capital requirement. In direct real estate, the calculation of this risk measurement is still rare even if it is now common to compute and disclose it in numerous other fields of finance. Indeed nowadays, financial institutions are facing the important task of estimating and controlling their exposure to market risk following a scope of new regulations such as Basel II, Basel III, NAIC or Solvency II. In this context, financial institutions use internal models for estimating their market risk. The purpose of this paper is to investigate the possibility to use Cornish-Fisher expansion to assess real estate value at risk. We show how Cornish-Fisher approximation can quickly give more accurate measurements than traditional methodologies. In addition, practitioners can find here a methodology to assess quickly value at risk without too many loss of relevancy due to normal hypothesis which is relaxed in our proposal.

After a review of literature on value at risk and of the existing methodologies, the paper describes the Cornish-Fisher expansion, the assumptions required to apply it and how the expansion is used to compute value at risk. Then, we apply the proposed model to a UK dataset index and compare the results obtained with those obtained with Gaussian assumption.

KEYWORDS: Value at Risk, Risk Measurements, Real Estate Finance, Cornish-Fisher Expansion, Risk Analysis, Risk Management

I. Introduction

a) Definition of VaR

Risk measurements have hugely changed since Markovitz (1952) developed his theory in the 50's. Standard deviation was then the risk measurement of an efficient portfolio. However this measurement was not relevant for one security only. Indeed, facing a unique security, the risk is computed using the covariance between this security and the market. Indeed, the standard deviation of a security is composed of risk that can be mitigated by diversification and by risk that cannot be diversified. Yet, only the risk that cannot be diversified might be remunerated. The risk theories that have followed the one of Markowitz have mainly been concentrated to the factors that determine the risk of a security and to the capital markets equilibrium. In fact, when considering a portfolio composed of N securities, Markowitz model requires the estimate of N variances and $\frac{N^2 - N}{2}$ covariances. When N becomes large, the estimation of the variance-covariance matrix becomes arduous and the possibilities of errors increase which can lead to misleading decision.

During the 60's, Sharpe (1964) developed the Capital Asset Pricing Model, a mono-factor model that considers the covariance between the security and the market as the only one risk factor. This risk is measured by the beta (β) and is called the systematic risk. It cannot be mitigated by diversification. On the contrary, the specific risk (the non-systematic one), inherent to the company, can be mitigated by diversification.

Then the Asset Pricing Theory was developed by Ross (1976) in the 70's. This model is a multifactor model and identifies the multidimensional effect of the risk. However, one of the weaknesses of this model is that it does not explain the factors that determine the return of the security. Value at risk did not appear before the late 80's. In 1987, the stock market crashed and the trigger event for a new risk measurement. This was the first major financial crisis where practitioners as well as academics were afraid about global bankruptcy of the entire system. The crash was so improbable to happen given standard statistical models that all the quants cast doubt and began to question the models. Many academics claimed that the crisis were recurring and ask for reconsidering the models. Taking into account extreme event had become obvious. The limitations of the traditional risk measurement were recognized and measuring the risk of fall of the value of the assets was becoming urgent.

The necessity to rely on a risk measurement that considers the entire distribution of return of a portfolio was obvious. In this context, throughout the 90's, a new risk measurement was built up: the Value at Risk with its acronym *VaR*[1]. VaR was developed and then adopted by practitioners and

[1] In our context and for all the paper, VaR is assumed to be computed for a static portfolio and with no change in its structure, no trading or arbitrage.

regulators. Jorion (2006) define VaR as follows, “VaR is a method of assessing risk that uses standard statistical techniques used routinely in other technical fields. Loosely, VaR summarizes the worst look over a target horizon that will not be exceeded with a given level of confidence”. In financial risk management, VaR is a measure of the risk of loss on a specific portfolio of financial assets (among which real estate asset). For a given portfolio, probability and time horizon, VaR is defined as a threshold value such that the probability that the mark-to-market loss on the portfolio over the given time horizon exceeds this value (assuming no trading in the portfolio) is the given probability level. On a portfolio value at time t, V_t, for a time horizon of one period of time and for a threshold α, this can be translated as:

$$\forall t, \mathrm{P}_t \left[\left(V_{t+1} - V_t \right) + VaR_{t,\alpha} < 0 \right] = \alpha$$

Or as well by considering a position X with its cumulative distribution function F_X and $q_\alpha(X)$ the lower fractile by:

$$VaR_\alpha(X) = -\sup\left\{ x \mid F_X(x) < \alpha \right\} = -q_\alpha(X)$$

Unlike the most widely adopted convention in the literature, some chose to count positively an effective loss:

$$VaR_\alpha(X) = \inf\left\{ x \mid F_X(x) > \alpha \right\}$$

This measurement is then become more and more popular to value the risk of institutional and individual portfolio. In particular, Value-at-Risk is seen as an easy to understand method for quantifying market risk. As of today, VaR is used by many regulators as the risk measurement reference, among others Basel I, Basel II, Solvency II and NAIC.

The worldwide adoption of the Basel II Accord in 1999 and near completion today (Basel III must be applied for 2019) gave further impulsion to VaR. Basle committee has required that banks compute periodically their VaR and maintain sufficient capital to pay the eventual losses projected by VaR. Unfortunately, there is not one measure of VaR because volatility, which is a fundamental component of VaR, is latent. Therefore, banks must use many VaR models to compute the range of their prospective losses. These computations might be complex because the distribution of returns is in general not known. Nowadays, the main uses of VaR in finance are – among others – risk management, risk analysis, financial control, financial reporting and computing regulatory capital. More recently, methodologies and risk measurements such as stress testing, expected shortfall, and tail value at risk have become more popular because they focus particularly on the expected severity of failure. VaR is thus slowly replacing standard deviation or volatility as the most widely used measure of risk. This has happened because of the need for a single risk measure for the calculation of capital adequacy limits for financial institutions such as banks or insurers. Value at Risk allows for regulators and banks management to put a single number on a predefined worst-case scenario (at a certain level of confidence). Nevertheless, VaR is not the maximum loss one may experience. It is in fact the lowest loss at this threshold α. Even if it is exact in theory, it works only for a specific confidence level. There is always a higher level of loss for a lower confidence level.

The three main traditional methods of calculating Value at Risk are:

1. The historical method
2. The variance-covariance method
3. The Monte Carlo method

The historical method involves taking empirical profit and loss history and ordering it, then assuming that history will repeat itself. The main benefit of the Historical method is that it does not require any assumptions about the nature of the distribution of returns. The major drawback is that this method implicitly assumes that the shape of future returns will be the same as those of the past. To make this approach statistically reliable, one need to ensure that sufficient number of observations is available and that they are representative of all possible states of the portfolio. Data must incorporate observations from both bull and bear markets. In real estate area, since we rarely have enough history (and more generally in almost all non listed market) the empirical method is not considered as accurate as either the parametric or simulation method.

The variance-covariance method (sometimes named parametric method) requires an assumption to be made about the statistical distribution (normal, log-normal etc.) from which the data is drawn. Parametric approaches are comparable to fitting curves through the data and then reading off the VaR from the fitted curve (unfortunately, for many sophisticated models, analytical solutions do not exist). The parametric VaR is one of the more popular methods. The attraction of parametric or analytic VaR is that relatively little information is needed to compute it. The main weakness is that the distribution chosen may not accurately reflect all possible states of the market and may under or overestimate the risk. This problem is particularly acute when using VaR to assess the risk of asymmetric distributions (in particular portfolio containing options). In such cases the higher statistical moments of skewness and kurtosis which contribute to more extreme losses (fat tails) need to be taken into account. So although some level of statistical sophistication is necessary, parametric methods exist for a wide variety of distributions.

The Monte-Carlo approach has become more and more popular in recent years. Mainly, this is due to the improvement of computer and software power. Monte Carlo methods rely on repeated random generation from a probability distribution of the inputs that are then used to compute the results of a model. Simulation based VaR generates thousand simulated scenarios drawn either from a parametric assumption about the shape of the distribution or by re-sampling the empirical history and generating enough data to be statistically significant. The Value at Risk is deducted by reading the desired percentile as in the historic calculation method.

Despite its popularity among practitioners, regulators but also academics, VaR is subject to many criticisms. It has been controversial since it moved from trading desks into the public eye in 1994. A common complaint among academics is that VaR is not subadditive. In fact, VaR does not systematically satisfy the property of convexity as illustrated by Danielson and al. (2005): the VaR of a combined portfolio can be larger than the sum of the VaR of its components. This was demonstrated by Artzner and al. (1999), excepting in some special cases (among which the normal distribution), VaR does not satisfy the subadditivity requirement for mathematical coherence. Also, assessing plausible losses is difficult using VaR theory. Losses can be extremely large and sometimes impossible to determine once one gets beyond the VaR point. From a practitioner point of view, VaR is more seen as the level of losses at which one stop trying to imagine what can happen next. Other academics such as Longin (2005) suggest taking an interest in extreme event (and therefore extreme value theory) only when appraising extreme risk such as Value at Risk. In real estate, the extreme value theory in the context of value at risk for listed real estate has been studied by Liow (2008).

b) Literature review

VaR has been the subject of a wide work among academics. All the methods that have been proposed to compute VaR or a quantile of the distribution have been subject to academics research quickly after the Value at Risk set-up in 1994. In particular, the researchers and academics from RiskMetrics have published a large number of papers on Value at Risk assessment. Among them Monte-Carlo simulation: Pritsker (1996); Johnson transformations: Zangari (1996a), Longerstaey (1996); Cornish-Fisher expansions: Zangari (1996b), Fallon (1996); the Solomon-Stephens approximation: Britton-Jones and Schaefer (1999); moment-based approximations motivated by the theory of estimating functions: Li (1999); saddle-point approximations: Feuerverger and Wong (2000); Fourier-inversion: Rouvinez (1997) or Albanese et al. (2000) and extreme value theory: Longin (2000).

Many works have concentrated on the best methodologies to use to compute value at risk. Pichler and Selitsch (1999) compare five different VaR-methods in the context of portfolio that includes options: Johnson transformations, Variance-Covariance, and three Cornish-Fisher-approximations for the second, fourth and sixth order. They conclude that the sixth order Cornish-Fisher approximation is the best approaches compared to the other approaches. The authors also suggest that methodologies that rely only on the first four moments are rather poor. Mina and Ulmer (1999) compare Johnson transformations, Fourier inversion, Cornish- Fisher approximations, and Monte Carlo simulation. The conclusion is the following: Johnson transformations are considered not to be a robust choice, Monte Carlo and Fourier inversion are robust and Cornish-Fisher is seen as extremely fast but a less robust than the two previous approaches in particular when the distribution is really far from the normal.

Feuerverger and Wong (2000) focus on when or when not to use Cornish-Fisher compared with Fourier inversion, saddle point methods, or Monte Carlo. This paper concludes by an extension of the methodology that includes higher-order terms. Jaschke (1999) concentrates on Cornish-Fisher properties and underlying assumptions in the context of Value at Risk with a particular focus on the monotony of the distribution function as well as convergence that are not guaranteed. Jaschke discusses how these assumptions make Cornish-Fisher seem undesirable and difficult to use. However, he demonstrates how – when the dataset fits the required assumptions – the accuracy of Cornich-Fisher expansion is generally more than sufficient in addition of being faster.

In fact, the proper use of Cornish-Fisher expansion should avoid two pitfalls: the existing domain of validity of the formula and confusing the skewness and kurtosis parameters of the formula with the actual skewness and kurtosis of the distribution. These assumptions have been discussed in Chernozhukov and al. (2010) and in Maillard (2012). Combining these two papers allows now to use this tool. Chernozhukov and al. (2010) propose a procedure called increasing rearrangement to monotonize Cornish-Fisher expansion. In addition ways to remedy the possible narrowness of the domain of validity are proposed. Maillard (2012) focus on the distinction between skewness and kurtosis parameters and actual values. These two papers have made our paper possible; indeed, this is following them that we are now able to compute real estate VaR using Cornish-Fisher.

In real estate field, VaR has been the subject of many papers. However, these papers mainly focus on listed real estate and not on direct real estate. Mainly VaR for listed real estate can rely on previous discussed methods for stocks or bonds. Zhou and Anderson (2010) concentrate on extreme risks and the behavior of REITs in abnormal market conditions. They found that no universal methodology can be recommended for VaR in listed real estate. Also the estimation of the risk for stock and REITs may require different methods. Liow (2008) makes use of extreme value theory to assess the VaR dynamics of ten major securitizes real estate markets. The use of extreme value theory allows the author to consider the stochastic behavior of the tail. Using this tool, the extreme market risk are better assess than with the traditional standard deviation measure and real estate forecasts are more accurate.

We did not find any paper that concentrates specifically on VaR in the context of direct real estate market. However numerous papers and research have concentrated on risk management and risk assessment in real estate. Gordon and Wai Kuen Tse (2003) consider VaR as a tool to measure leverage risk in the case of a real estate portfolio. The level of debt of a real estate portfolio is a traditional issue in real estate finance. The paper shows how the use of VaR allows a better assessment of risk. In particular the traditional risk adjusted measure (Sharpe ratio, Treynor's and Jensen's alpha) suffer from the leverage paradox. Leverage adds risk along with the potential for higher returns per unit of higher risk. Therefore the ratio risk/return does not change noticeably and is not an accurate tool to measure the risk inherent to the level of debt. On the contrary, VaR is good tool for leverage risk. Brown and

Young (2011) focus their work on a new way to measure real estate investment risk. They begin by refusing the assumption of normally distributed returns that flaw the forecasts and decisions. The nature of risk and how it should be measured is discussed. Interestingly, the value at risk is not retained and the expected shortfall is more recommended. The authors focus therefore their work on spectral measures which is their final recommendation.

From our knowledge, the use of Cornish-Fisher expansion to determine VaR in real estate has not been subject of large literature. Lee and Higgins (2009) make use of Cornish-Fisher expansion in the real estate context. They argue that Sharpe performance formula neglects two important characteristics of real estate returns: non-normality and autocorrelation. They apply Cornish Fisher expansion to adjust the Sharpe ratio performance to the non-normality.

c) Motivation

One of the major issues – if not the worst – to assess direct real estate VaR is the lack of data as far as statistics are concerned. On a micro-market the data is somehow available but on a macro view of the market, we face the difficulty to deal with small database. This is particularly true in commercial real estate where institutional investors mostly invest their money. In this sense, the real estate market is comparable to the private equity market where indexes are built on small number of transaction. Real estate property index attempts to aggregate real estate market information to provide a representation of the underlying real estate performance. However this is generally done on a monthly basis in the best case, on a quarterly or semi-annually basis sometimes and generally on a yearly basis. This is largely linked to the sector, residential where many transactions can be observed exhibits generally monthly index and commercial real estate (office, activities, shopping centers…) face more difficulties to deliver frequent indexes. To determine the VaR of a real estate portfolio at a threshold of 0.5% (as requested by Solvency II framework) using the historical approach, a minimum of 200 values is needed (which represent 17 years on a monthly index basis). And even with that, the VaR is the minimum of the series. With a monthly index, this requires a minimum of 17 years of data. Alternatively, estimating the distribution of a series of return (mandatory with variance - covariance and Monte Carlo methodologies) requires a certain amount of data and we one more time face the real estate lack of data issue. This is therefore incredibly difficult to assess value at risk in presence of small database. In addition, facing numerous indexes, the choice of the best index can also become an arduous task (valuation based index, transaction based index…) as underlined by Kovac and Lee (2008). This is why alternative methods that do not rely too much on strong assumptions must be envisaged.

Non normality is a fact of life as far as the distribution of property prices or returns are concerned. Real estate returns are known for displaying non normal return. This has long been demonstrated by Myer and Webb (1994) or Young and Graff (1995). More recent researches such as Young, Lee and Devaney (2006) or Alcock, Glascock and Steiner (2012) have shown how real estate returns usually exhibit non normal returns. Real estate returns are generally left skewed and exhibits fat tails. The table 1 presents some basic statistics about real estate returns over some countries extracted from IPD database. The table shows that real estate returns generally have left skewed returns and fat tails (South Africa and Germany are two exceptions)

Country	South Africa	Netherlands	Germany	Japan	Italy	France (office)	Australia	USA	UK
Database length	1995-2010	1995-2010	1989-2010	2003-2010	2003-2010	1986-2010	Dec 84-Dec 10	Mar 99-Dec 10	Dec 86-Dec 10
Index publication	Annual	Annual	Annual	Annual	Annual	Annual	Quarterly	Quarterly	Monthly
Mean	15,9%	9,9%	4,6%	5,2%	6,6%	8,6%	10,4%	8,2%	9,5%
Volatility	7,4%	4,5%	2,1%	7,1%	3,5%	9,5%	9,0%	11,6%	11,9%
skewness	0,76	-0,80	-0,01	-0,34	-0,83	-0,28	-0,09	-2,10	-0,92
Excess kurtosis	-0,39	0,36	-0,24	-1,04	-0,72	-0,94	0,27	4,73	1,03

Index: Total return all properties (except France: office)

Table 1: Real estate returns basic statistics (IPD)

These facts have to be considered when determining real estate VaR. The case of Solvency II regulation (European regulation for insurers) is particularly interesting. They have based the capital requirement on VaR estimation. They propose either the use of a standard model or of an internal built model. The standard model for real estate value at risk has estimated a required capital of 25% for real estate investment. This calculation was made on IPD UK all properties total return index. Indeed this is the one of the only one reliable commercial monthly index in Europe[2]. However, the committee itself recognizes the non-normality of real estate return but did not try to estimate the real estate required capital taking into account the observed non-normality. Following these observations, we seek VaR methodologies that consider the non normality of real estate returns in VaR computation. This is exactly what Cornish Fisher do. Cornish Fisher makes us able to consider moments of order higher than two and therefore to consider non normality of distributions. The Cornish-Fisher approximation transforms the quantile of a normal law in a realized value where skewness and excess kurtosis are not equal to zero.

In this article we concentrate particularly on direct real estate value at risk and propose the use of Cornish Fisher expansion to improve traditional model.

The remainder of the paper is organized as follow. Section 2 introduces the Cornish-Fischer expansion and discusses some technical points. Section 3 carries out an implementation of the model. Section 4 discusses some limitations of the model and is followed by a conclusion in the final section.

[2] CEIOPS-DOC-40/09
- 3.160: “One of the most challenging factors of this specific calibration is the lack of long time series across most European markets”
- 3.169 & 3.171: “All distributions of property returns are characterized by long left fat-tails and excess kurtosis signifying disparity from normal distribution”

II. Variance-Covariance Value at Risk and Cornish Fisher adjustment

The distribution used to estimate the VaR of a portfolio is determined from the distribution of the returns of the portfolio or of the sector indexes. When dealing with real estate, the question which arises, due to the non traded nature of properties, is whether the low numbers of data in the index make it relevant to use and whether this index represents the investor's portfolio. As developed previously, the main problem faced by real estate practitioners and academics is the size of database. Either you invest in listed real estate and in this case, real estate is quoted daily and enough data are available to compute the VaR of your portfolio, or you invest in direct real estate and you have to deal with smaller database. We present first the main used method to determine value at risk in presence of small database and then the Cornish Fisher adjustment.

2.1) VaR with a Normal assumption: Variance-covariance approach

If the returns are supposed to be normal, it is possible to estimate the fractile of the distribution corresponding to the threshold. Therefore the random variable *X* that represents the value of the portfolio follow:

$$X \sim N\left(\mu, \sigma^2\right)$$

The random variable can therefore be written as a standard normal variable ε such as:

$$X = \mu + \varepsilon\sigma$$

If z_α is the threshold of probability for the risk measurement, it can therefore be rewritten:

$$X = \mu + z_\alpha \sigma$$

The VaR is thus computed as follow:

$$VaR = E\left(X\right) - U_\alpha = \mu - \left(\mu + \alpha\sigma\right) = -\alpha\sigma$$

With U_α, the fractile associated to the threshold α.

2.2) VaR with quasi Normal assumption: Cornish-Fisher expansion

According to Stuart and al. (1999), a large number of distributions tend toward the normal when the number of observations *n* tends toward infinity. However for small sample, normal distribution is generally not very suitable. In particular in real estate, the absence of centralized market price and the low number of transactions among the markets lead to get small sample for direct real estate. Hence, normality assumption seems to be a too strong assumption. The idea is to correct the discrepancies arising from normal quantiles. Basically this expansion is an approximate relation between the percentiles of a distribution and its moments. This approximation is based on the Taylor series. It relies on the moments of a distribution that deviate from the normal law to determine the percentiles of this distribution.

The Cornish Fisher expansion has been developed by Cornish and Fisher (1937). This expansion is a formula to approximate fractile of a random variable based only its first few cumulants. The cumulants of a random variable *X* are conceptually similar to its moments. They are defined as those value κ_r such that the identity

$$\exp\left(\sum_{r=1}^{\infty}\frac{\kappa_r t^r}{r!}\right)=\sum_{r=0}^{\infty}\frac{E(X^r)t^r}{r!}$$

holds for all *t*. If a distribution is fitted by making the first moments of the fitted distributions agreed, it is, in principle, possible to calculate quantiles of the fitted distribution and to regard these as approximations to the corresponding quantiles of the actual distribution. <So we have estimators of the actual quantiles which are functions of these moments. Usually these functions are very complicated and not easily expressible in explicit form. However, in the case of independent and identically distributed random variables it is possible to obtain explicit expansions for standardized quantiles as functions of corresponding quantiles of the unit normal distributions. In these expansions the terms are polynomial functions of the appropriate unit normal quantile, with coefficients that are functions of the moment-ratios of the distribution. This approach leads to an analytic approximation of the quantile as long as the moments of the distribution are known. As an example, the Cornish-Fisher approximation taking into account the first sixth moments yields to:

$$
\begin{aligned}
\alpha - quantile \approx \quad & z_\alpha + \frac{1}{6}\left(z_\alpha^2 - 1\right)E\left(X^3\right) \\
& + \frac{1}{24}\left(z_\alpha^3 - 3z_\alpha\right)E\left(X^4\right) - \frac{1}{36}\left(2z_\alpha^3 - 5z_\alpha\right)E\left(X^3\right)^2 \\
& + \frac{1}{120}\left(z_\alpha^4 - 6z_\alpha^2 + 3\right)E\left(X^5\right) - \frac{1}{24}\left(z_\alpha^4 - 5z_\alpha^2 + 2\right)E\left(X^3\right)\ E\left(X^4\right) \\
& + \frac{1}{324}\left(12z_\alpha^4 - 53U_\alpha^2 + 17\right)E\left(X^3\right)^3 \\
& + \frac{1}{720}\left(z_\alpha^5 - 10z_\alpha^3 + 15z_\alpha\right)E\left(X^6\right) \\
& - \frac{1}{180}\left(2z_\alpha^5 - 17z_\alpha^3 + 21z_\alpha\right)E\left(X^3\right)\ E\left(X^5\right) \\
& - \frac{1}{384}\left(3z_\alpha^5 - 24z_\alpha^3 + 29z_\alpha\right)E\left(X^4\right)^2 \\
& + \frac{1}{288}\left(14z_\alpha^5 - 103z_\alpha^3 + 107z_\alpha\right)E\left(X^3\right)^2 E\left(X^4\right) \\
& - \frac{1}{7776}\left(252z_\alpha^5 - 1688z_\alpha^3 + 1511z_\alpha\right)E\left(X^3\right)^4 + \cdots
\end{aligned}
$$

where $E\left(X^n\right)$ denotes the moment of order n and z_α is the percentile corresponding to the N(0,1). Details on the expansions are reported to Johnson and Kotz (1970).

Taking into account the kurtosis excess and neglecting all non significant terms, the Cornish-Fisher expansion using the first four moments of the distribution gives then:

$$q_\alpha \cong U_\alpha + \frac{1}{6}\left(U_\alpha^2 - 1\right)E\left(X^3\right) + \frac{1}{24}\left(U_\alpha^3 - 3U_\alpha\right)E\left(X^4\right) - \frac{1}{36}\left(2U_\alpha^3 - 5U_\alpha\right)E\left(X^3\right)^2$$

where $E\left(X^3\right)$ is the asymmetric coefficient (S) and $E\left(X^4\right)$ the excess kurtosis (K-3) of a distribution, if X is centered and reduced. This can be rewritten

$$q_\alpha \cong z_\alpha + \frac{1}{6}\left(z_\alpha^2 - 1\right)S + \frac{1}{24}\left(z_\alpha^3 - 3z_\alpha\right)(K-3) - \frac{1}{36}\left(2z_\alpha^3 - 5z_\alpha\right)S^2$$

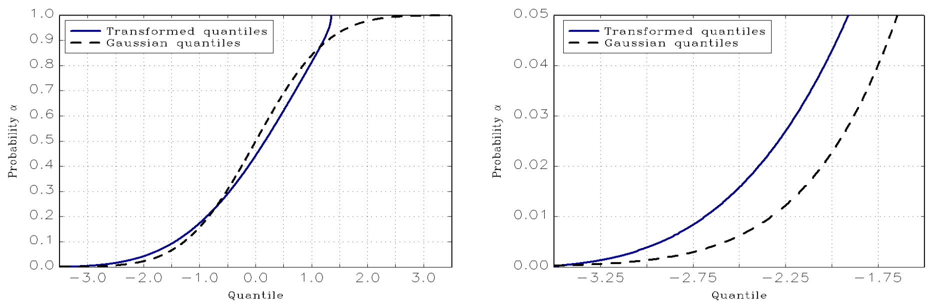
The VaR is then given by:

$$VaR = \mu + q_\alpha \sigma$$

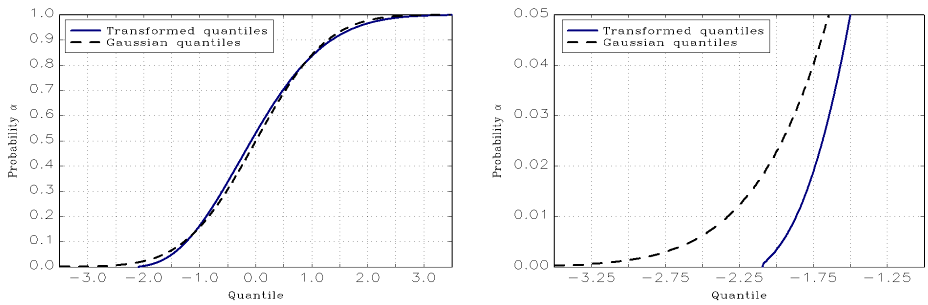
The VaR calculated using Cornish Fisher expansion seeks to modify the multiple associated to the normal law in order to take into account the moments of order higher than two of the return distribution[3]. We illustrate below the effect of non normality on quantile below.

[3] Instead of determining the value of this multiple associated to the Cornish Fisher expansion, many financial institution increase the multiple associated the normal law in order to take into account the moments of order 3 and 4. As an example, the multiple associated to a threshold of 5% is -1,645 for the normal law. In order to consider the leptokurtosis of the returns distribution some financial institutions use a multiple equal to 2 or 3. This methodology is not scientific and Cornish Fisher expansion has to be preferred.

For a left skewed distribution, the smallest quantiles are lower (higher VaR) than the gaussian ones. It is the contrary for a right skewed distribution. The first point is illustrated by Figure 1a where a skewness of -1 leads to $q_{0.05}$ = -1.9103 while $z_{0.05}$ = -1.6449. We have as well $q_{0.001}$ = -3.3049 and $z_{0.001}$ = -3.0902. On Figure 1b, S=0.5, we get for instance $q_{0.05}$ = -1.4980.



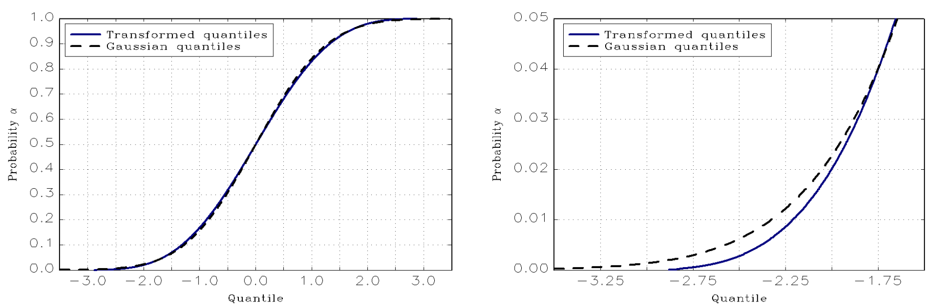
a) Left skewed distribution: *S*=-1 and *K* = 3



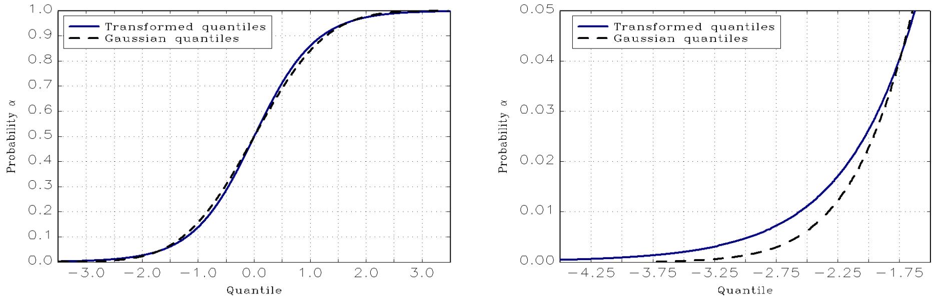
b) Right skewed distribution: *S*=0.5 and *K* = 3

Figure 1: Cornish-Fisher and normal quantiles according to the skewness coefficient

A leptokurtic (platokurtic) distribution implies lower (higher) smallest quantiles as shown on Figure 2a (2b). The highest correction for the VaR will happen in the case of a left skewed and leptokurtic distribution as illustrated on Figure 2bis.



a) Platokurtocic distribution: K = 2.5 and S=0



b) Leptokurtocic distribution: K = 4 and S=0

Figure 2: Cornish-Fisher and normal quantiles according to the kurtosis coefficient

The use of Cornish-Fisher should avoid a pitfall: the existing domain of validity of the formula. There is a domain of validity for the use of the Cornish-Fisher expansion (see Maillard, 2012). To be valid, this transformation has to be bijective. It is a necessary and sufficient condition. It implies that the derivative of q_α relative to z_α is non null. If the transformation is not bijective, the order in the quantiles of the distribution would not be conserved. This can be written

$$\frac{S^2}{9} - 4\left(\frac{K}{8} - \frac{S^2}{6}\right)\left(1 - \frac{K}{8} - \frac{5S^2}{36}\right) \leq 0$$

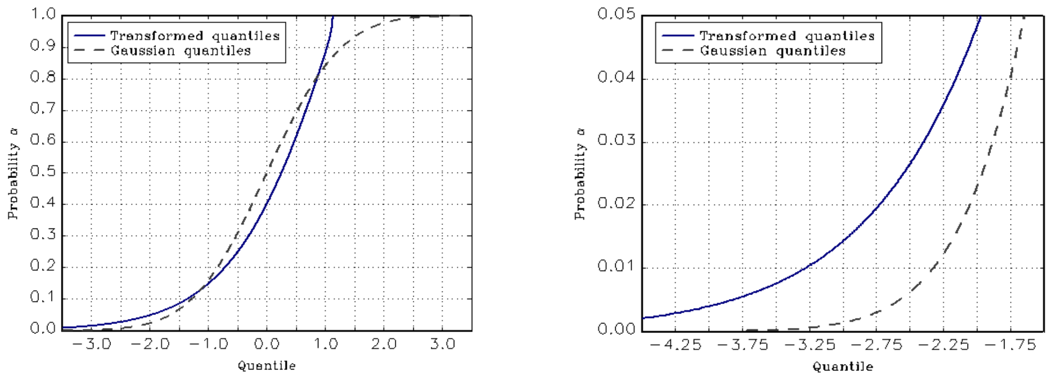


Figure 2bis: Cornish-Fisher quantiles, left skewed and leptokurtic distribution
S=-1.5; *K*=6

A consequence of the non bijection of the Cornish-Fisher expansion is the quantile function is not monotonic, which violates an obvious monotonicity requirement. This arises because the polynomials which are in the transformation are non monotonic. Chernozhukov, Fernàndez-Val and Galichon (2010) propose a procedure to restore the monotonicity called the rearrangement. As they mention and demonstrate, the rearrangement necessarily brings the non-monotone approximations closer to the true monotone target function. Figure 3 illustrates this procedure for a skewness of 0.8 and a kurtosis of 2 (an excess of kurtosis equal to -1). The entire cdf function estimation is presented in appendices figure A.3. These parameters correspond to a platokurtic and right skewed distribution function.

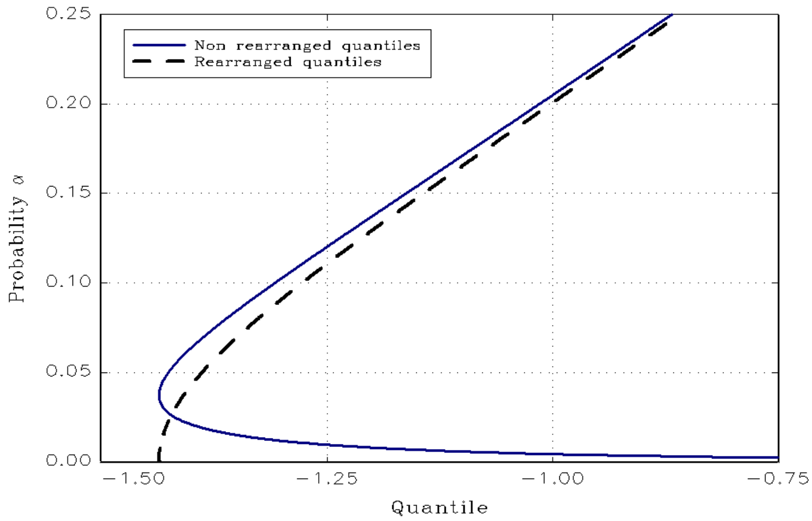


Figure 3: Rearrangement procedure (α <0.25)

As the kurtosis coefficient has to be greater than 3 in order to have the bijection (this is a necessary condition), the quantile function is not monotonic in this example (see Figure 3). The discreapency from the two quantile functions appears for the smallest probabilities which are the most important for the V.a.R. computation. The non rearranged quantile function could be more erratic as the one presented below (see for instance Figure 1 in Chernozhukov, Fernàndez-Val and Galichon, 2010). Let us denote $\tilde{q}_\alpha$ the corrected quantile of level α. At 0.1%, $\tilde{q}_\alpha = -1.4$ while z_α is clearly biased and is equal to -0.3.

III. Application

We study the UK real estate return from December 1987 to December 2010, which leads to 277 observations. We study the database and then determine the quantiles and value at risk at a threshold of 5, 1, 0.5 and 0.1%. The 0.5% is the threshold required by solvency II regulation. The values are annualized monthly returns.

The index and the corresponding returns are presented on Figure 4. It exhibits clearly both the 90' overproduction crises and the subprime periods.

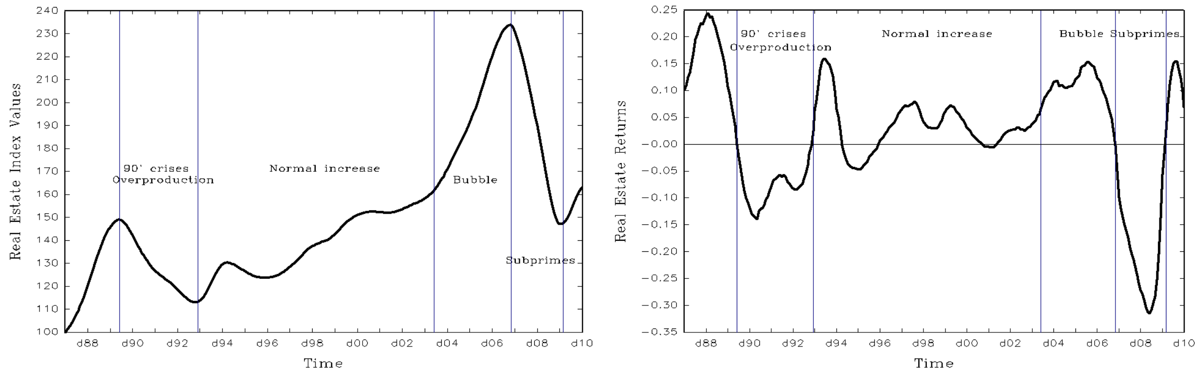


Figure 4: Real estate index and returns from December 1987 to December 2010

The distributions of the returns differ significantly across the periods as illustrated on Figure 5. We cut the database in periods representing various state of the cycle. This corresponds to the first 10-years period, the middle 10-years and the last 10-years periods that run from December 2000 to December 2010.

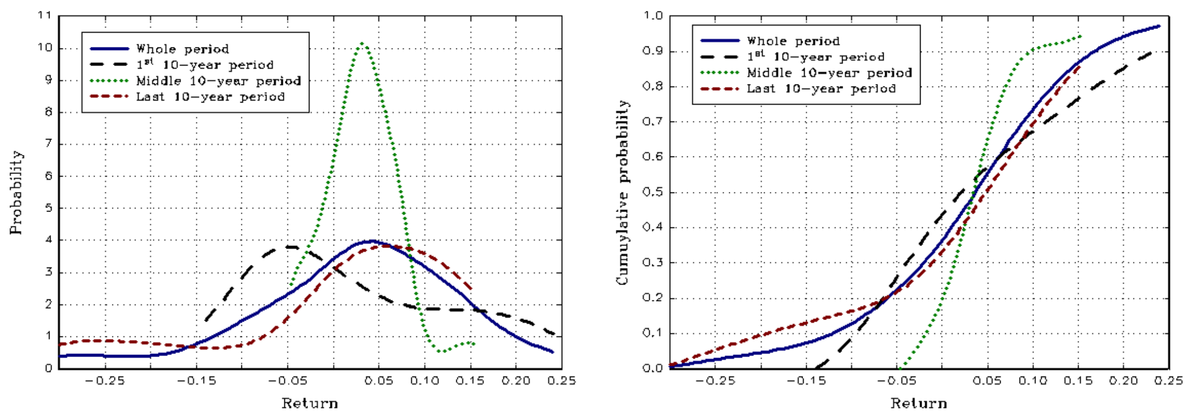


Figure 5: Real estate returns pdf and cdf according to the period

If the middle 10-year period distribution is more concentrated (from June 1994 to June 3004), the last ones is clearly left skewed with high negative returns (subprime crises). These negative returns will lead to a different analysis of the risk as shown in Table 1 when analyzing the descriptive statistics presented in Table 1.

Periods	Mean	s.d.	Min	max	*S*	*K*	Q1	Q2	Q3
1987-12 / 2010-12	0.0215	0.1131	-0.3154	0.2439	-0.7559	3.7935	-0.0436	0.0310	0.1010
1987-12 / 1997-12	0.0253	0.1090	-0.1396	0.2439	0.5058	2.0611	-0.0625	-0.0028	0.1091
1994-06 / 2004-06	0.0297	0.0417	-0.0468	0.1580	0.4661	3.9611	0.0017	0.0298	0.0555
1990-12 / 2010-12	0.0076	0.1310	-0.3154	0.1543	**-1.1743**	3.3457	-0.0053	0.0281	0.1081

Table 2: Real estate monthly returns pdf and cdf according to the period

As mentioned previously, a long dataset is recommended to compute the VaR, here we decide to compute our results on a 15 years basis (180 returns) in order to take into account more than one cycle and in order to obtain results that are not too erratic. Taking this recommendation into account will lead us using a 15-year rolling period to compute the moments and the distribution of the returns. Each of the 97 periods contains 180 observations. The estimation of the distribution in December 2002 is made using the returns from December 1987 to December 2002.

Moreover, Figure 6 gives the 95% bootstrap confident interval of the mean and standard deviation and mainly, those of the skewness and the kurtosis. The mean and the standard deviation are not table at all. The returns are increasing from December 1999 to the subprime crises (during the normal increase and the bubble), and are falling down. The evolution of the standard deviation seems to be opposite. At the contrary the evolutions of *S* and *K* are more dichotomous: i) nearly stable around 0 and 3 before the subprime crises, ii) highly left skewed (*S*<0) and leptokurtic left skewed (*K*>3) after. More precisely, let us remark that until December 2001, the distribution is platokurtic (the kurtosis coefficients are significantly different from 3). Moreover form December 1997 to December 2007, the skewness is either null or significantly positive (right skewed, due to high returns as during the bubble period for instance).

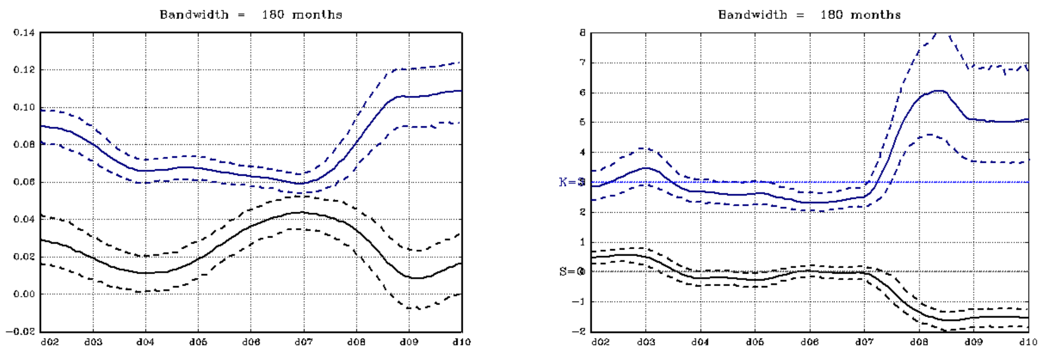


Figure 6: mean, strandard deviation, skewness and kurtosis according to the 15-year period

Given *S* and *K* for each 15-year rolling period, we are able to compute the Cornish-Fisher correction of the quantiles. Figure 7 presents the results for the 5%, 1%, 0.5% and 0.1% quantiles of the real estate returns distribution. The dotted lines correspond to the quantiles obtained without using the correction proposed by Chernozhukov, Fernàndez-Val and Galichon (2010). The correction is noticeable during the first months of the computation and (from December 2002), during the bubble period and after the crash. In particular, it can be noticed how important this correction is when the threshold decrease. This correction is more relevant when the lower quantiles (for the 0.5% threshold of Solvency II quantile for instance). For each of the four analyzed quantiles, the “true” one is often lower than the

Gaussian ones until the subprime crises. This is particularly obvious for the 0.5% and 0.1% quantiles, which the more interesting quantiles for VaR computation. Except for the 1% quantile, we get

- In December 2002: the quantile is less negative than the Gaussian one (less risky);
- From December 2002, the quantile decreases in order to reach the Gaussian one, at the end of 2004;
- From the end of 2004 to December 2007 (bubble period), we get again less negative quantiles than the Gaussian one;
- In 2007-2008, a fall of the quantile value leads to a more risky situation;
- In 2008-2009, the quantile level increases a bit but still remains below the Gaussian value. (more pronounced for the lowest quantiles).

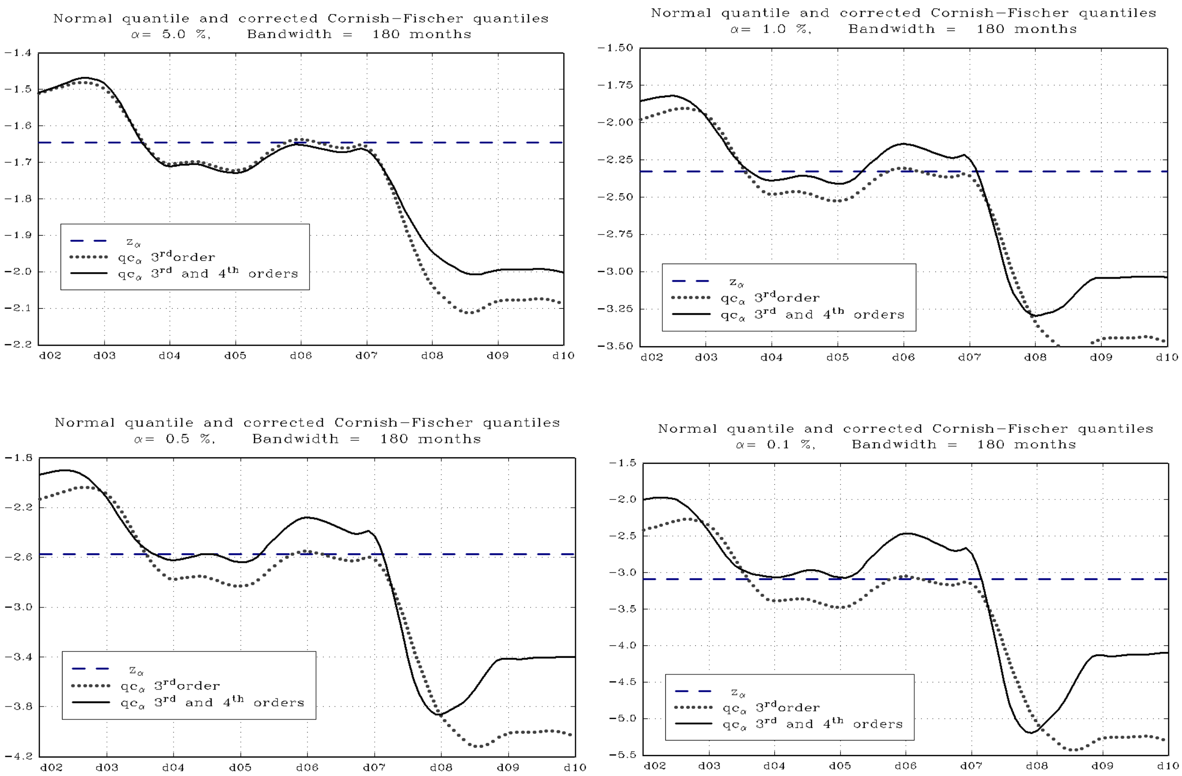


Figure 7: 5%, 1%, 0.5% and 0.1% quantiles of the real estate returns distribution according to the 15-years period

We now compute the VaR given the same threshold with a 15 years window. The interesting point is to compare our results with the ones obtain by the regulator. Solvency II regulation requires a 25% of required capital for real estate investment. Their valuation was based on UK all properties total return database with a threshold of 0.5%. Our results at a 0.5% are concordant with those of the regulators. The interesting point is to notice that the valuation of regulators is close to the one obtained with the

Gaussian assumption. However taking into account moments of order higher than 2 leads to an higher VaR and therefore to higher required capital. This results show in particular how essential it is to consider skewness and kurtosis to properly assess real estate value at risk. The Gaussian assumption is really not adequate.

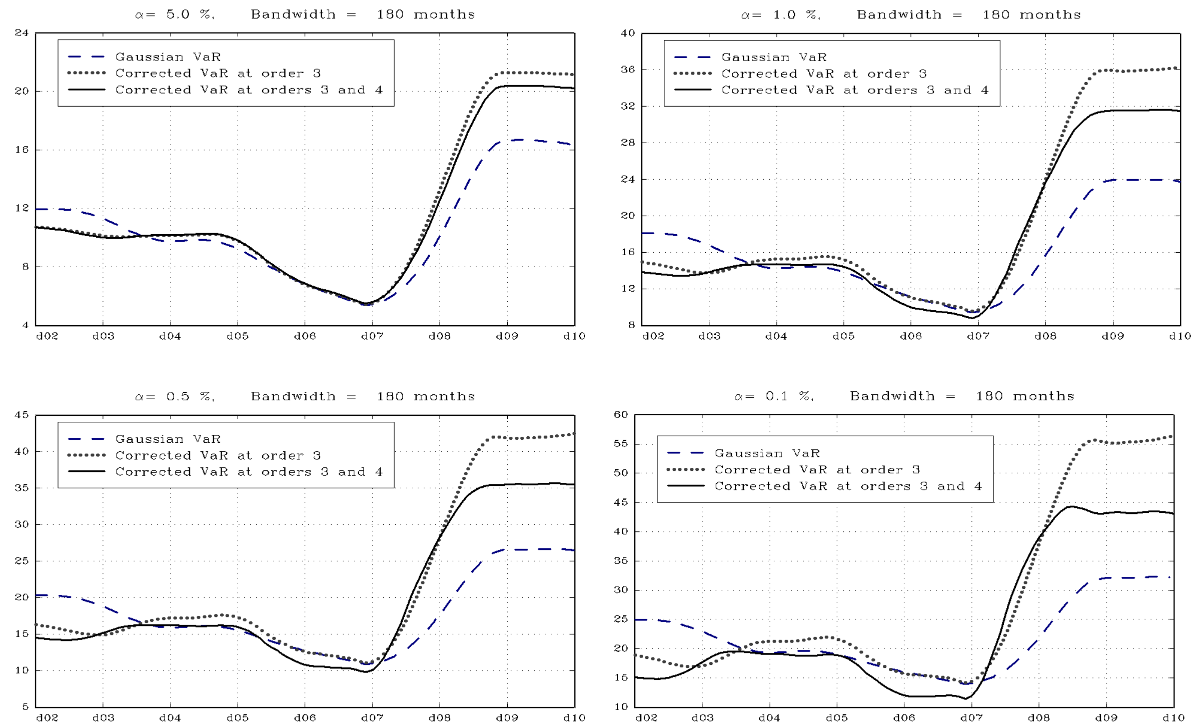


Figure 8: 5%, 1%, 0.5% and 0.1% Gaussian and corrected Cornish-Fisher VaR in base 100 according to the 15-years period

We focus in Figure 9 on the importance of the database length. The length of the period is key when assessing VaR. The VaR is the maximum loss at a threshold level for predefined length in time. However, facing small database issues lead to make choices about length of time.

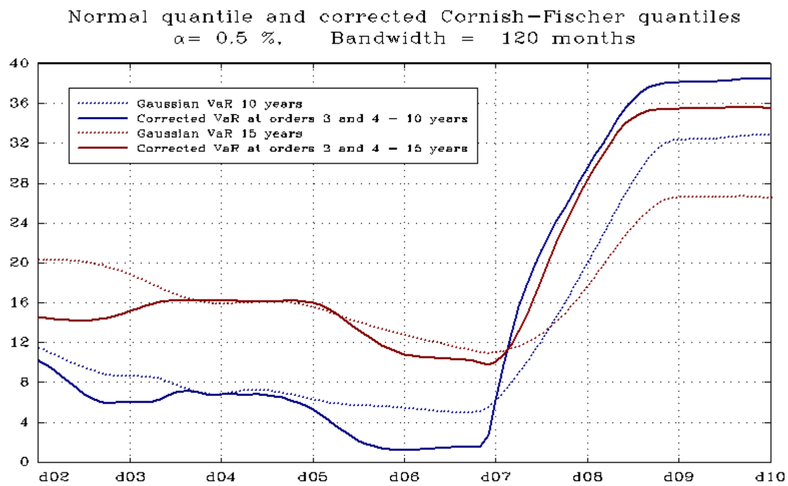


Figure 9: Gaussian VaR and Cornish-Fisher corrected VaR for 10 and 15 years

Like all the methodologies proposed for VaR estimation, Cornish-Fisher methods suffer from certain limitation. We discuss briefly the limitation of the methodology.

IV. Limitation

The use of Cornish-Fisher expansion for VaR shows certain limitations. Some assumptions such as quasi-Gaussian innovations, monotony and quadratic approximation can be questioned. Britton-Jones and Schaefer (1999) show that quadratic approximations can lead to large errors when computing VaR. In addition the Taylor-approximation holds only locally which can be a huge issue when modeling extreme events. Embrechts et al. (1999) also show that the Gaussian framework does not allow to model joint extremal events. Chernozhukov and al. (2010) demonstrate how rearrangement can solve the monotonic assumption’s issue that can lead to important shortcomings. Despite these valid critiques on Cornish Fisher expansion model, there are good reasons for real estate practitioners, banks or insurances to implement it alongside other models in particular in the real estate context where the dataset are modest. When you face lack of data, no methodologies provide correct outputs. The statistical assumption of quasi-Gaussian distribution can explain a part of the errors but the lack of data also explains part of errors. The way in which this VaR model can be assessed statistically is by comparing its performance with the historical model and determining the number of times VaR exceeded with what is expected for the model. However, on more time, there is insufficient historical data to perform a backtest and so a qualitative assessment has to be done instead.

Obviously VaR is a risk measurement that only takes into account the probability of being below the threshold level. It does not does not consider the values below this level or their average. In addition, VaR is a poor measure for asymmetric distribution of returns. It can also exhibit convexity issues. This is why other risk measurement has been proposed. Among them expected shortfall as defined by Acerbi and al. (2001) (also called conditional VaR: CVaR) or the TailVaR in Artzner and al. (1999). Their application in real estate finance will be the subject of further research and in particular the application of Cornish-Fisher in their context will be the subject of a future paper.

V. Conclusion

Based on the IPD monthly capital return from December 1987 to December 2010 data, the research shows that the UK direct property data has substantial departures from normality. The research focuses on moments of order higher than two and proposes a way to incorporate them in VaR assessment and therefore to get over the classic shortcut of Gaussian hypothesis. In particular the use of rearrangement procedure as demonstrated by Chernozhukov, Fernandez-Val and Galichon (2010) allows to overpasses the non monotonic issue when the transformation is not bijective. This way, we are able to apply Cornish-Fisher expansion to real estate returns in order to determine more accurately (taking into account skewness and kurtosis) the Value at Risk of a portfolio.

Our results show that methodologies that do not consider skewness and kurtosis to compute VaR lead to a bad estimation of the risk. In particular, in presence of skewed return and fat tails which is the case in real estate market, we obtain an undervaluation of the VaR which can lead to non adequate capital requirements. This research calculates a number of VaR and quantile to examine the effects thresholds have on the risk measure performance. The results indicate that Cornish-Fisher methodology is more accurate when the threshold is rather low.

In terms of professional application, the expansion combined with rearrangement procedure could be of the interest of professional. The expansion can be used and the methodology replicated by professionals to determine the VaR of their portfolio or to determine their required capital using an internal model. In particular, professionals invested in secured properties only can be interested in applying this methodology to an index representing the market in which they invest. They might demonstrate that the risk taken for investing in secured properties is below the risk of the market and therefore their required capital is below the one of the standard model.

The methodology is particularly robust for distributions that are non-normal and can therefore apply to hedge fund industry or private equity.

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Appendices :

1. We show that since 2007, the UK real estate returns cannot be considered as normal. However, the returns were normal from 2003 to mid-2007 on a 15-years window. Figure A.1 presents thep-values of the Jarque Berra normality test on a 15-years window.

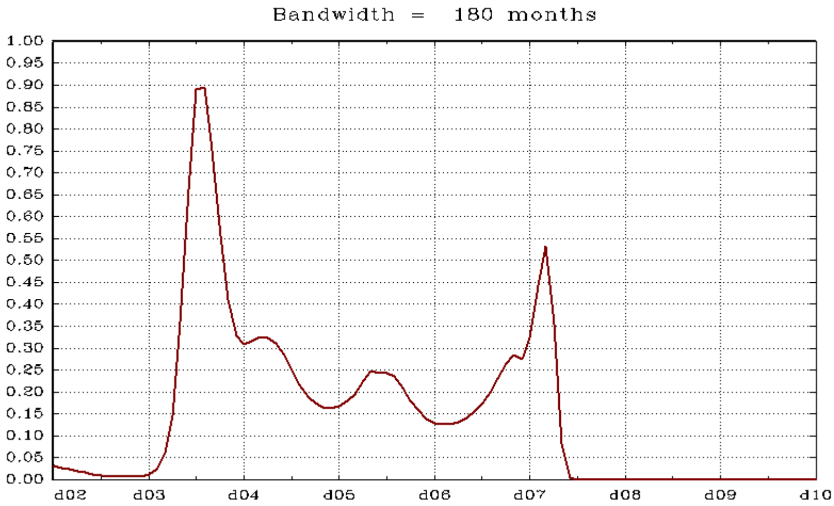

Figure A.1.a: p-values of the real estate returns for the Jarque Berra normality test

For information, the same results are displayed on a 10-years windows basis. It is clearly more erratic.

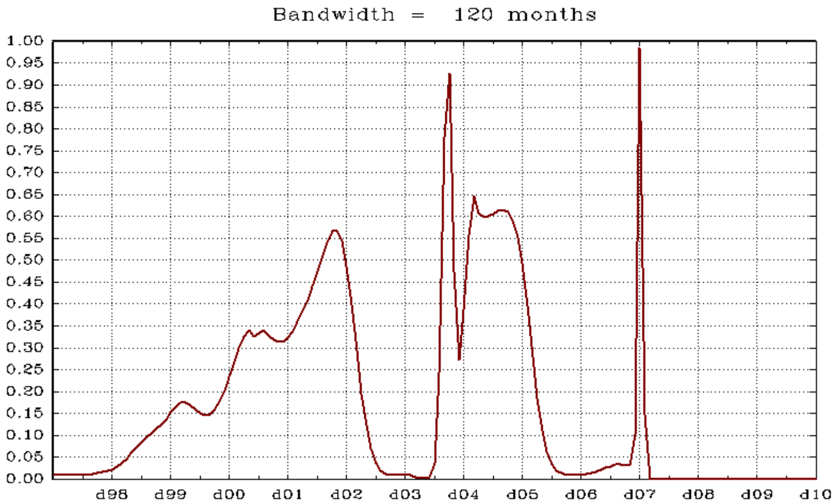

Figure A.1.b: p-values of the real estate returns for the Jarque Berra normality test

2. We present below another correction of the quantile when the tails of the distribution are fat.

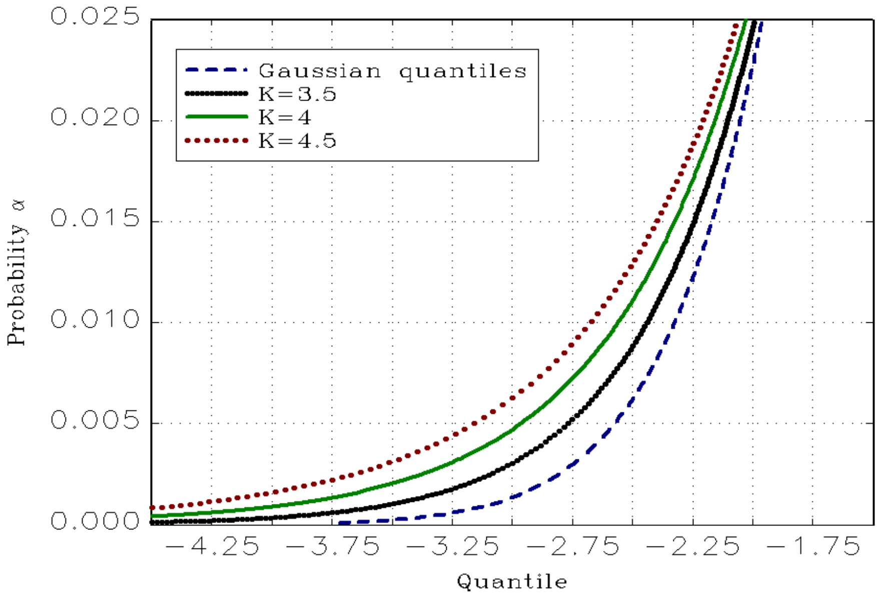

Figure A.2: Quantile estimation in presence of tails

3. The rearrangement procedure allows to get over the traditional pitfalls faced by Cornish Fisher expansion. This is illustrated in the figure A.3. Figure 3 in the paper is an enlargement of the A.3 figure on the lowest qunatiles.

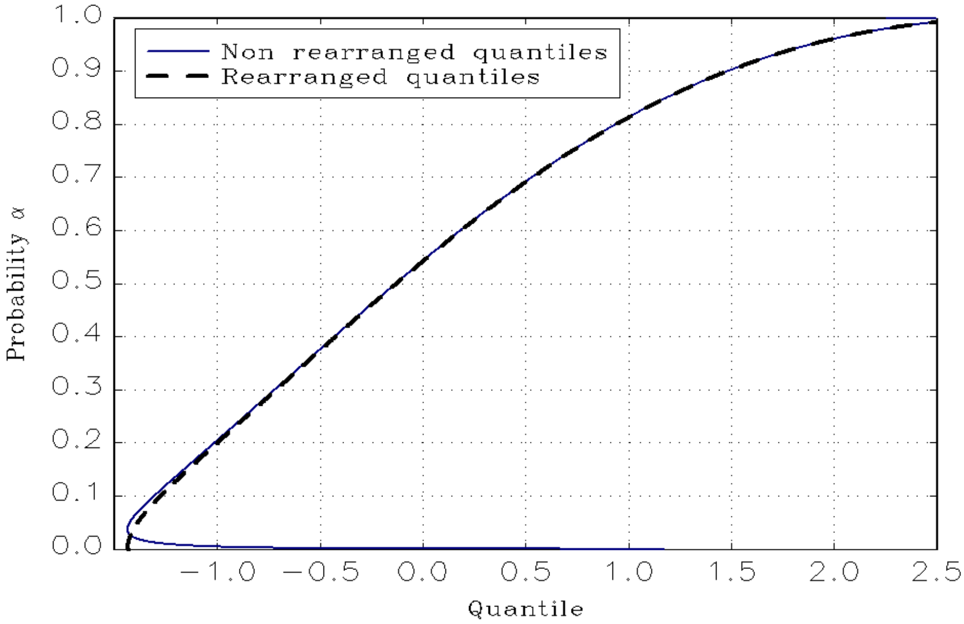

Figure A.3: Rearrangement procedure illustration (α <0.25)